Poisson Clusters and Unique Factorization

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[joint work with Milen Yakimov]

Quick cluster algebra sketch (geometric type; coeffs $\in$ field)

$K \subset F = K(y_1, \ldots, y_N)$ = rational function field

<u>clusters</u> = transcendence bases for F/K

initial cluster $= (y_1, \ldots, y_N)$

$[1, N] \supseteq$ <u>ex</u> = set of <u>exchangeable</u> indices (*others are <u>frozen</u>*)

$M_{N \times \underline{\mathrm{ex}}}(\mathbb{Z}) \ni B =$ <u>exchange matrix</u> (*with some conditions*)

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cluster $(y_1, \ldots, y_N) \rightsquigarrow$ *cluster* $(y_1, \ldots, y_{k-1}, y'_k, y_{k+1}, \ldots, y_N)$

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Iterate mutations in all $\underline{\mathrm{ex}}$ directions

cluster algebra $:=$ K-subalgebra of F generated by $\bigcup$ all clusters

from iterated mutations, together with

y_k^{-1} for k in some set $\underline{\mathrm{inv}} \subseteq [1, N] \setminus \underline{\mathrm{ex}}$

upper cluster algebra $:=$

$\bigcap$ of $K[\, z_i^{\pm 1} \mid i \in \underline{\mathrm{ex}} \sqcup \underline{\mathrm{inv}}\,][\, z_i \mid i \notin \underline{\mathrm{ex}} \sqcup \underline{\mathrm{inv}}\,]$

for original cluster and one-step mutations in all $\underline{\mathrm{ex}}$ directions upper cluster algebra :=

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Laurent Phenomenon [Fomin-Zelevinsky]

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Some known cluster algebras : homogeneous coordinate rings of

- Grassmannians $Gr(m, n)$ [Scott]
- partial flag varieties in semisimple algebraic groups type ADE [Geiß-Leclerc-Schröer]

Some known upper cluster algebras : coordinate rings of

- double Bruhat cells in semisimple algebraic groups / $\mathbb{C}$ [Berenstein-Fomin-Zelevinsky]

Assume $\mathrm{char}(K) = 0$ *from now on* $\qquad [K = \text{base field}]$

Poisson algebra $=$ a commutative algebra R with Lie bracket

$\{-,-\} : R \times R \longrightarrow R$ such that all $\{r,-\}$ are derivations

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E.G. $\mathcal{O}(M_{m,n}(K))$ with the standard Sklyanin bracket :

$$\{X_{ij}, X_{il}\} = X_{ij}X_{il} \qquad (j < l)$$
$$\{X_{ij}, X_{kj}\} = X_{ij}X_{kj} \qquad (i < k)$$
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and coordinate rings of Poisson subvarieties of $M_{m,n}(K)$,

such as $GL_n(K)$, double Bruhat cells of $GL_n(K)$

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- a cluster $(z_1, \ldots, z_N)$ is log-canonical if $\{z_i, z_j\} \in K z_i z_j \;\; \forall\, i, j$
- the cluster structure on A is Poisson-compatible iff

  all clusters are log-canonical

Poisson polynomial algebra (*Poisson version of skew poly ring*)

$R = K[x_1][x_2;\sigma_2,\delta_2]_p \cdots [x_N;\sigma_N,\delta_N]_p$:

a polynomial ring $K[x_1,\dots,x_N]$ with Poisson bracket $\ni$

$$\{x_k, r\} = \sigma_k(r)x_k + \delta_k(r) \quad \text{for all} \quad r \in K[x_1,\dots,x_{k-1}]$$

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R ($\uparrow$) is a Poisson-nilpotent algebra iff $\exists$ K-torus $H = (K^\times)^r$ $\ni$

- H acts rationally on R by Poisson automorphisms
- All x_k are H-eigenvectors
- All δ_k are locally nilpotent
- Each σ_k given by action of $h_k \in \mathrm{Lie}\, H$, with $h_k \cdot x_k \neq 0$

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E.G. $R = \mathcal{O}(M_{m,n}(K))$ with Sklyanin bracket,

$H = (K^\times)^{m+n}$, $(\alpha_1, \dots, \alpha_m, \beta_1, \dots, \beta_n) \cdot X_{ij} = \alpha_i \beta_j X_{ij}$

In a Poisson algebra R :

- Poisson ideal $I \lhd R$: $\{R, I\} \subseteq I$
- Poisson-normal element $c \in R$: $\{c, R\} \subseteq cR$
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Consequence : All Poisson-normal H-eigenvectors in R are products of units and Poisson-prime H-eigenvectors, unique up to ordering and associates.

Initial clusters :

Thm 2. [Yakimov-K.G.] Let $R = K[x_1, \ldots, x_N]$ be a Poisson-nilpotent algebra.

$\exists$ Poisson-prime H-eigenvectors $y_k \in K[x_1, \ldots, x_k]$ $\forall\ k$ $\ni$

- All Poisson-prime H-eigenvectors in $K[x_1, \ldots, x_k]$ are among the scalar multiples of $y_1, \ldots, y_k$.
- $(y_1, \ldots, y_N)$ is log-canonical $\big(\ \{y_k, y_l\} \in K y_k y_l\ \big)$.
- $K[y_1, \ldots, y_N] \ \subseteq\ R\ \subseteq\ K[y_1^{\pm 1}, \ldots, y_N^{\pm 1}]$.

A Poisson-nilpotent algebra $R = K[x_1, \ldots, x_N]$ is symmetric if :

- $\delta_k(x_j) \in K[x_{j+1}, \ldots, x_{k-1}] \quad \forall\ k > j$
- $R = K[x_N, x_{N-1}, \ldots, x_1]$ is Poisson-nilpotent with
    - The same torus H
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If R is a symmetric Poisson-nilpotent algebra, then $\forall\, \tau \in \Xi_N$:

- $R = K[x_{\tau(1)}, x_{\tau(2)}, \ldots, x_{\tau(N)}]$ is Poisson-nilpotent.
- The corresponding y-elements from Theorem 2 form a log-canonical cluster $(y_{\tau,1}, y_{\tau,2}, \ldots, y_{\tau,N})$.

Thm 3. [Yakimov-K.G.] Let $R = K[x_1, \ldots, x_N]$ be a symmetric Poisson-nilpotent algebra (*with mild conditions on scalars*).

Set ex $:= \{ k \in [1, N] \mid y_k$ is *not* Poisson-prime in $R \}$.

- R is a Poisson-compatible cluster algebra.
- $R =$ the corresponding upper cluster algebra.
- R is generated by the cluster variables $y_{\tau,k}$ for $\tau \in \Xi_N$ and $k \in [1, N]$.
- Also true for $R[\, y_k^{-1} \mid k \in \underline{\text{inv}}\,]$, any $\underline{\text{inv}} \subseteq [1, N] \setminus \underline{\text{ex}}$.

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Application : The coord rings of double Bruhat cells in semisimple algebraic groups / $\mathbb{C}$ are Poisson-compatible cluster algebras (*with the inital cluster data of* [*Berenstein-Fomin-Zelevinsky*]).

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Specialize: Take $m = n$ and

$$Y := \{\, [\, 1, \dots, i \mid n+1-i, \dots, n \,] \mid 1 \le i \le n \,\} \cup \{\, [\, n+1-i, \dots, n \mid 1, \dots, i \,] \mid 1 \le i \le n \,\}$$

Then $R[\, y^{-1} \mid y \in Y \,] =$ the coordinate ring of the open double Bruhat cell in $GL_n(K)$, and $R[\, y^{-1} \mid y \in Y \,]$ is a Poisson-compatible cluster algebra.