Biclosed sets in representation theory

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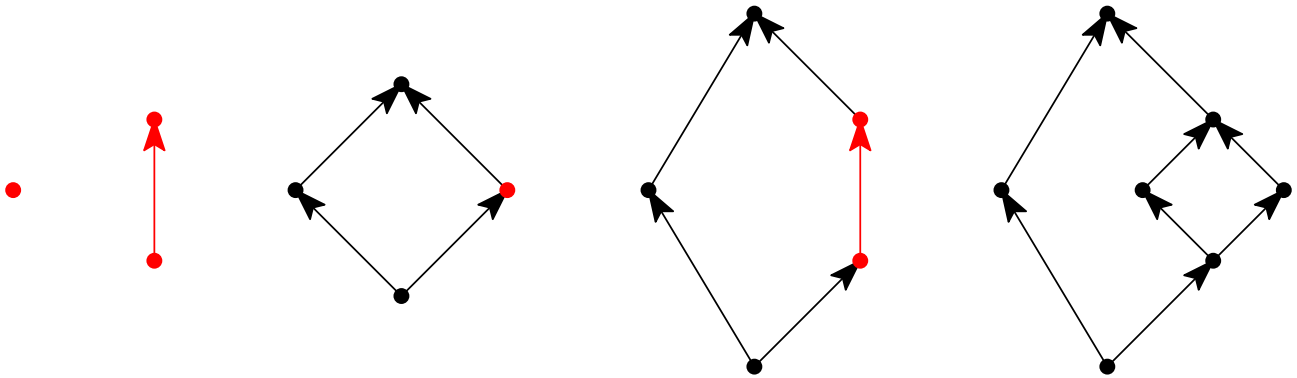
Outline

- Congruence-uniform lattices L
- Biclosed sets
- Shard intersection order $\Psi(L)$
- Applications

A finite lattice L is **congruence-uniform** if it may be constructed by a sequence of interval doublings from the one element lattice.



Theorem (Demonet–Iyama–Reading–Reiten–Thomas, 2017)

The lattice of torsion classes of a representation finite algebra Λ, denoted $\mathrm{tors}(\Lambda)$, is congruence-uniform.

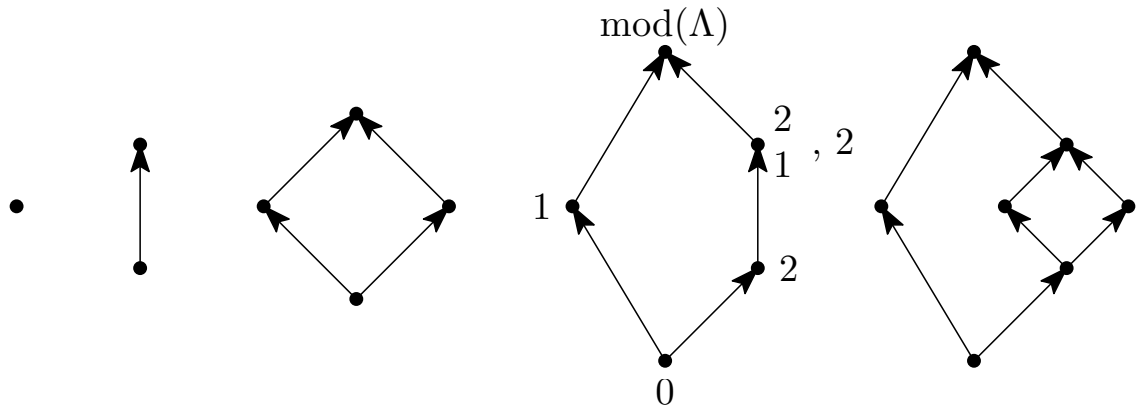
A full, additive subcategory $\mathcal{T} \subset \mathrm{mod}(\Lambda)$ is a **torsion class** if the following hold:

- if $X \twoheadrightarrow Z$ and $X \in \mathcal{T}$, then $Z \in \mathcal{T}$, and
- if $0 \to X \to Z \to Y \to 0$ and $X, Y \in \mathcal{T}$, then $Z \in \mathcal{T}$.

Example

$\Lambda = \Bbbk(1 \longleftarrow 2)$

A finite lattice L is **congruence-uniform** if it may be constructed by a sequence of interval doublings from the one element lattice.



Theorem (Demonet–Iyama–Reading–Reiten–Thomas, 2017)

The lattice of torsion classes of a representation finite algebra Λ, denoted tors(Λ), is congruence-uniform.

A full, additive subcategory $\mathcal{T} \subset \mathrm{mod}(\Lambda)$ is a **torsion class** if the following hold:

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Example

$\Lambda = \Bbbk(1 \longleftarrow 2)$

Goal: Find other examples of congruence-uniform lattices appearing in representation theory.

Assume $\Lambda = \Bbbk Q/I$ is a **gentle algebra**. The indecomposable Λ-modules are **string** and **band modules** [Wald–Waschbusch, 1985].

A word $w = \gamma_d^{\epsilon_d} \cdots \gamma_1^{\epsilon_1}$ with $\gamma_i \in Q_1$ and $\epsilon_i \in \{\pm 1\}$ is a **string** in Λ if

- w defines an irredundant walk in Q, and
- w and $w^{-1} := \gamma_1^{-\epsilon_1} \cdots \gamma_d^{-\epsilon_d}$ do not contain a subpath in I.

Example

Sequence $w = \delta\alpha^{-1}\gamma^{-1}\beta^{-1}$ is a string in $\Lambda = \Bbbk Q/\langle \alpha\beta \rangle$.

$$\begin{tikzcd}
 & 1 \arrow[rr, "\delta"] \arrow[dr, "\alpha"] & & 4 \\
3 \arrow[ur, "\beta"] & & 2 \arrow[ll, "\gamma"] &
\end{tikzcd}$$

Q

$$\begin{tikzcd}
 & \Bbbk^2 \arrow[rr, shift left, "\begin{bmatrix} 1 & 0 \end{bmatrix}"] \arrow[rr, shift right, "\begin{bmatrix} 1 & 0 \end{bmatrix}"'] \arrow[dr] & & \Bbbk \\
\Bbbk \arrow[ur, "\begin{bmatrix} 0 \\ 1 \end{bmatrix}"] & & \Bbbk \arrow[ll, "1"] &
\end{tikzcd}$$

$M(w)$

string module

$$\begin{tikzcd}
 & 1 \arrow[dl] \arrow[dr] & & \\
4 & & 2 \arrow[dr] & \\
 & & & 3 \arrow[dr] \\
 & & & & 1
\end{tikzcd}$$

$\mathrm{Str}(\Lambda) := \{\text{strings in } \Lambda\}$

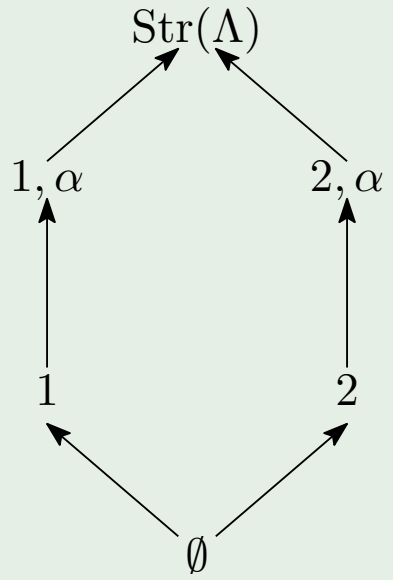
$B \subset \mathrm{Str}(\Lambda)$ is **closed** if
$w_1, w_2 \in B$, $w_1\alpha^{\pm 1}w_2 \in \mathrm{Str}(\Lambda)$ for some $\alpha \in Q_1 \implies w_1\alpha^{\pm 1}w_2 \in B$

$$\mathrm{Bic}(\Lambda) := \{\textbf{biclosed sets}\} = \{B \subset \mathrm{Str}(\Lambda) : B, \mathrm{Str}(\Lambda)\backslash B \text{ are closed}\}$$

Example

$\Lambda = \Bbbk(1 \xleftarrow{\alpha} 2)$
$\mathrm{Str}(\Lambda) = \{1, 2, \alpha\}$

$\mathrm{Bic}(1 \xleftarrow{\alpha} 2) =$



Exercise

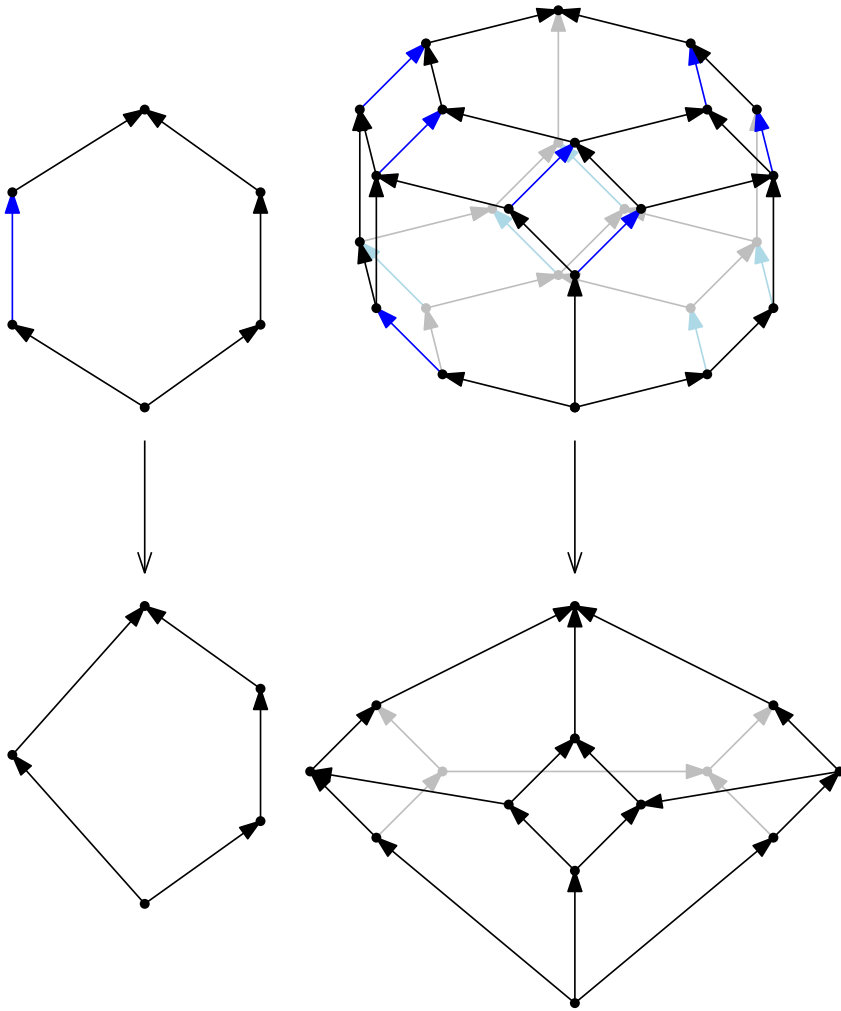
The weak order on $\mathfrak{S}_{n+1}$ is isomorphic to $Bic(1 \leftarrow 2 \leftarrow \cdots \leftarrow n)$.

Theorem (Palu–Pilaud–Plamondon, 2017)

If Λ is gentle and $|Str(\Lambda)| < \infty$, then $Bic(\Lambda)$ is congruence-uniform.

Why study biclosed sets?

- The poset of finite biclosed sets of positive roots is isomorphic to the weak order on the corresponding Coxeter group [Kostant, 1961].
- The biclosed sets in this talk were introduced to understand the lattice structure on a **lattice quotient** of biclosed sets [McConville, 2015], [Garver–McConville, 2015].
- A geometric analogue of our lattice quotient map had already been studied for generalized permutahedra and generalized associahedra [Hohlweg–Lange–Thomas, 2007].



Lattice quotient maps

$\Lambda = \Bbbk(\ 1 \underset{\beta}{\overset{\alpha}{\rightleftarrows}} 2\ )/\langle \alpha\beta, \beta\alpha \rangle$ (any gentle algebra can be expressed as $\Bbbk Q/I$ where $I = \langle \alpha_1\beta_1, \ldots, \alpha_k\beta_k \rangle$)

$$\Pi(\Lambda) = \Bbbk(\ 1 \begin{array}{c} \xleftarrow{\alpha^*} \\ \xrightarrow{\alpha} \\ \xleftarrow{\beta} \\ \xrightarrow[\beta^*]{} \end{array} 2\ )/\langle \alpha\beta, \beta\alpha, \beta^*\alpha^*, \alpha^*\beta^* \rangle$$

$\widetilde{\mathrm{Str}}(\Lambda) := \{$strings $\widetilde{w}$ in $\Pi(\Lambda)$ mapping to $\mathrm{Str}(\Lambda)$ via $(\alpha^*)^{\pm 1} \mapsto \alpha^{\pm 1}\}$

Example

String $\alpha^{-1}\alpha^* \in \mathrm{Str}(\Pi(\Lambda))$, but $\alpha^{-1}\alpha^* \notin \widetilde{\mathrm{Str}}(\Lambda)$ since $\alpha^{-1}\alpha \notin \mathrm{Str}(\Lambda)$.

Theorem (G.–McConville–Mousavand)

If Λ is a gentle algebra with no ***ouroboros****, then there is an isomorphism of posets $Bic(\Lambda) \cong \{\mathcal{T} \cap \mathcal{M}_\Lambda : \mathcal{T} \in tors(\Pi(\Lambda))\}$ where*

$$\mathcal{M}_\Lambda := add\left(\bigoplus_{\widetilde{w} \in \widetilde{Str}(\Lambda)} M(\widetilde{w}) \right).$$

Theorem (G.–McConville–Mousavand)

If Λ is a gentle algebra with no ***ouroboros****, then there is an isomorphism of posets $Bic(\Lambda) \cong \{\mathcal{T} \cap \mathcal{M}_\Lambda : \mathcal{T} \in tors(\Pi(\Lambda))\}$. We refer to the categories $\mathcal{T} \cap \mathcal{M}_\Lambda$ as* ***torsion shadows*** *and denote the lattice of torsion shadows by $torshad(\Lambda)$.*

A string $w \in \mathrm{Str}(\Lambda)$ is an **ouroboros** if it starts and ends at the same vertex.

Example

String $\beta\alpha$ is an ouroboros in

$$\Lambda = \Bbbk\left(1 \underset{\beta}{\overset{\alpha}{\rightleftarrows}} 2 \right)/\langle \alpha\beta \rangle.$$

Proposition (G.–McConville–Mousavand)

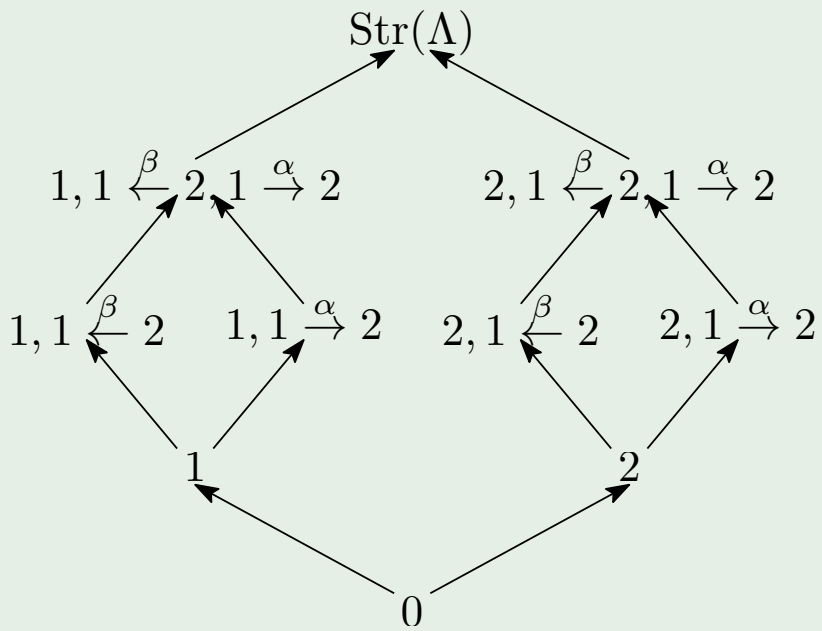
A gentle algebra Λ has no ouroboros if and only if every indecomposable Λ-module M is a ***brick*** *(i.e., $End_\Lambda(M)$ is a division algebra).*

The isomorphism is given by $B \longmapsto \mathrm{add}(\bigoplus_{\widetilde{w}} M(\widetilde{w}))$ where

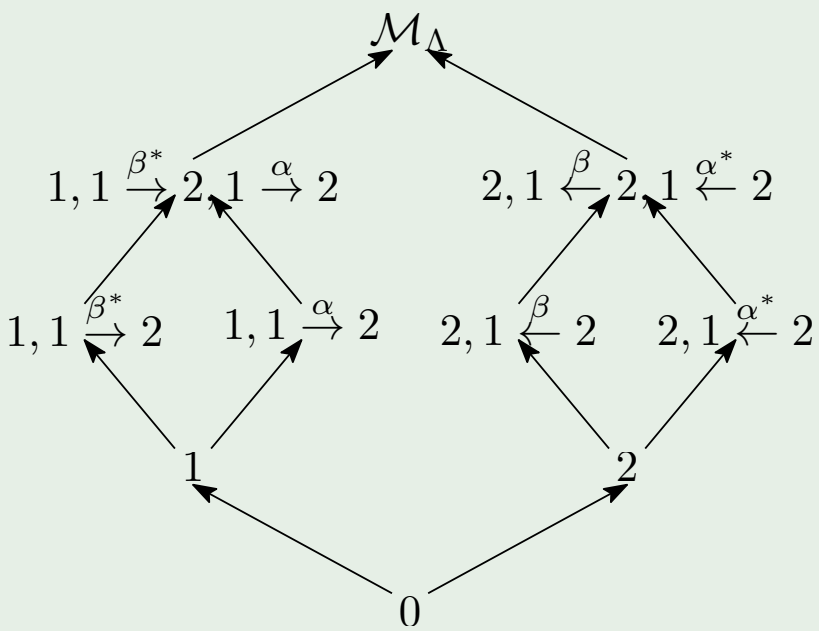
- $\widetilde{w} \in \widetilde{\mathrm{Str}}(\Lambda)$ and
- if $M(\widetilde{w}) \twoheadrightarrow M(\widetilde{u})$, then $\widetilde{u}$ maps to $u \in B$ via $(\alpha^*)^{\pm 1} \mapsto \alpha^{\pm 1}$.

Example

$$\Lambda = \Bbbk\left(1 \underset{\beta}{\overset{\alpha}{\rightleftarrows}} 2 \right)/\langle \alpha\beta, \beta\alpha \rangle$$

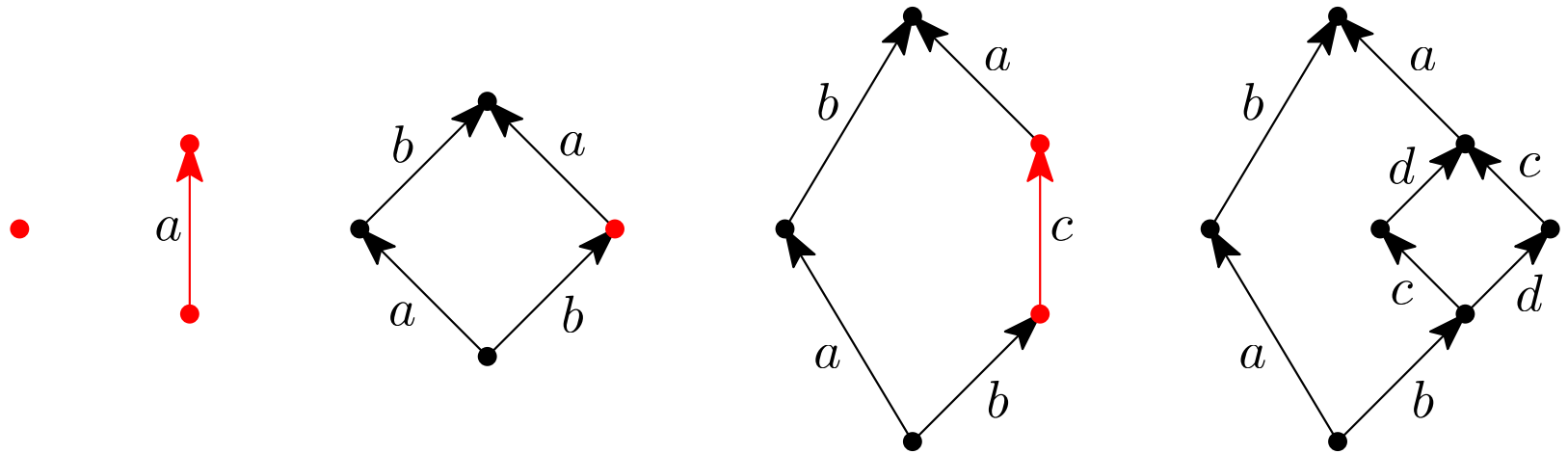


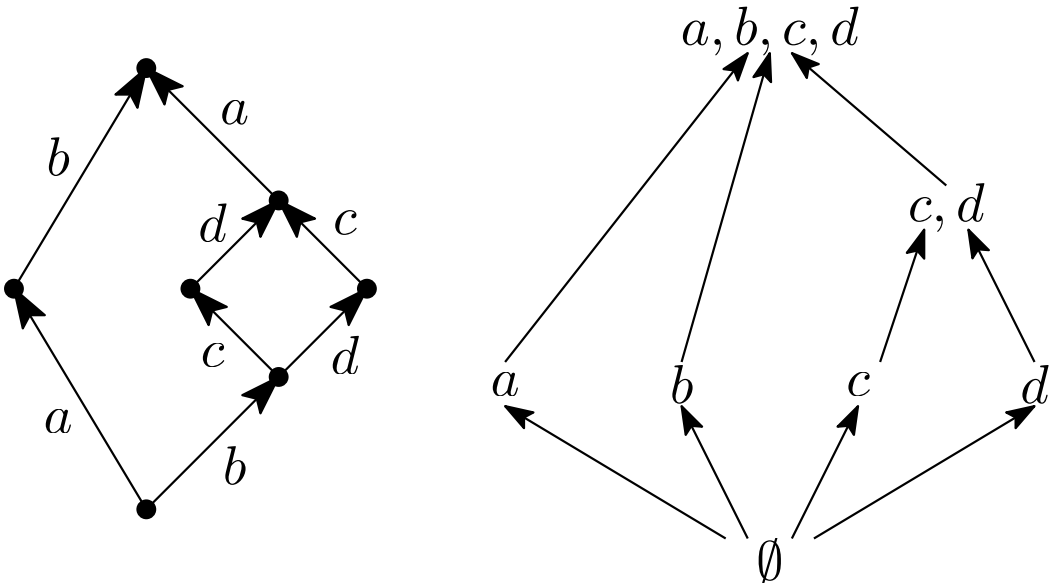
Bic(Λ)



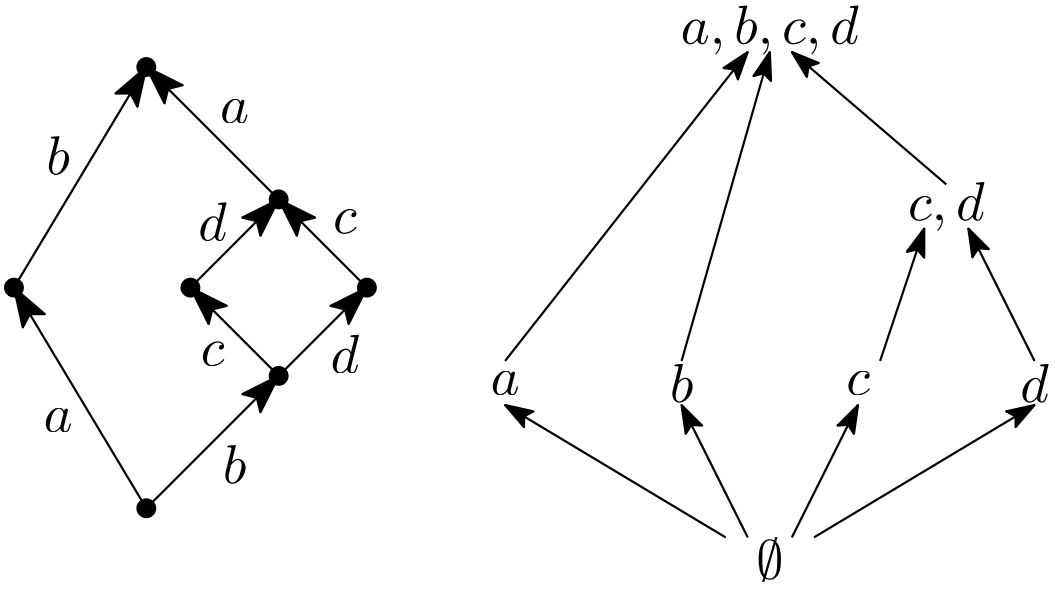
torshad(Λ)

Congruence-uniformity is equivalent to a function $\lambda : \mathrm{Cov}(L) \to P$ with certain properties. Say λ is a **CU-labeling** of L.





L $\qquad$ $\Psi(L)$



L $\Psi(L)$

- L a congruence-uniform lattice,
- $\lambda : \mathrm{Cov}(L) \to P$ a CU-labeling,
- $x \in L$ and covers $y_1, \ldots y_k \in L$,
- $\lambda_{\downarrow}(x) := \{\lambda(y_i, x)\}_{i=1}^{k}$

The **shard intersection order** of L, denoted $\Psi(L)$, is the collection of sets

$$\psi(x) := \left\{ \text{labels on covering relations in } \left[\bigwedge_{i=1}^{k} y_i, x \right] \right\}$$

partially ordered by inclusion.

Goal: Describe $\Psi(\mathrm{torshad}(\Lambda))$ using the representation theory of Λ. Here the CU-labeling is given by

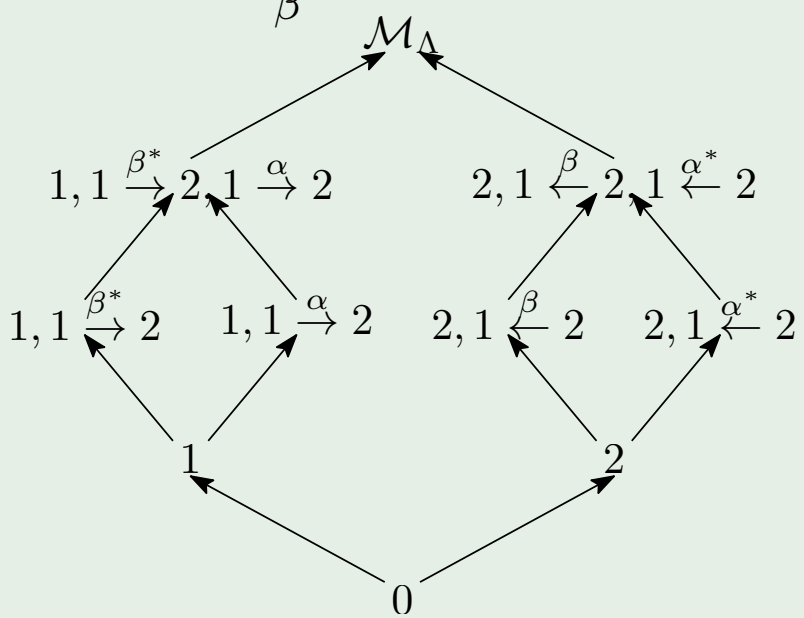
$$\begin{array}{rcl} \lambda : \mathrm{Cov}(\mathrm{torshad}(\Lambda)) & \longrightarrow & \mathcal{M}_\Lambda \\ (\mathcal{T}_1 \cap \mathcal{M}_\Lambda, \mathcal{T}_2 \cap \mathcal{M}_\Lambda) & \longmapsto & M(\widetilde{w}) \end{array}$$

where $\widetilde{w} \in \mathrm{Str}(\Pi(\Lambda))$ is the unique string satisfying

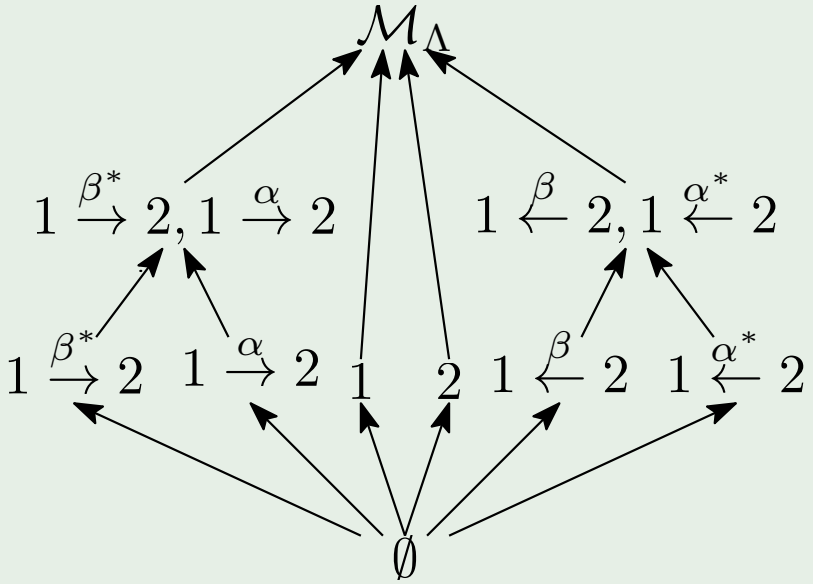
$$\mathrm{ind}(\mathcal{T}_2 \cap \mathcal{M}_\Lambda) = \mathrm{ind}(\mathcal{T}_1 \cap \mathcal{M}_\Lambda) \cup \{M(\widetilde{w})\}$$

Example

$\Lambda = \Bbbk(\ 1 \underset{\beta}{\overset{\alpha}{\rightleftarrows}} 2\ )/\langle \alpha\beta, \beta\alpha \rangle$



torshad(Λ)



$\Psi(\mathrm{torshad}(\Lambda))$

Recall that a full, additive subcategory $\mathcal{W} \subset \mathrm{mod}(\Lambda)$ is **wide** if

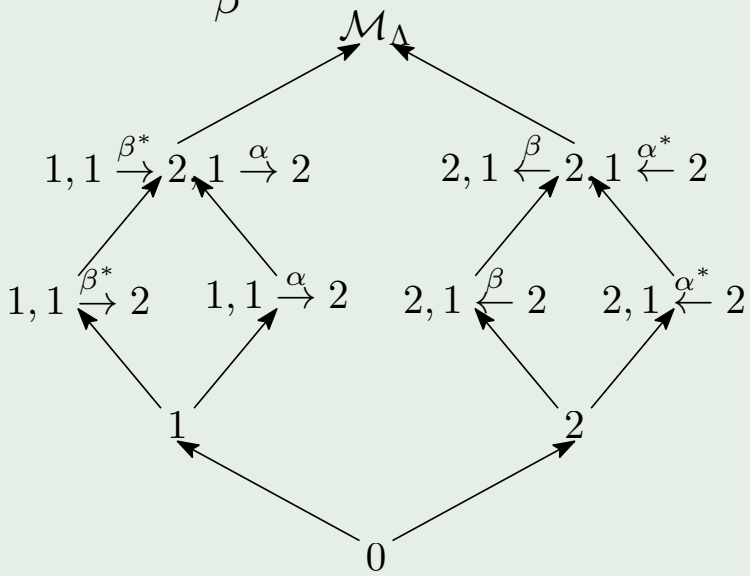
- $\mathcal{W}$ is abelian and
- if $0 \to X \to Z \to Y \to 0$ with $X, Y \in \mathcal{W}$, then $Z \in \mathcal{W}$.

Theorem (G.–McConville–Mousavand)

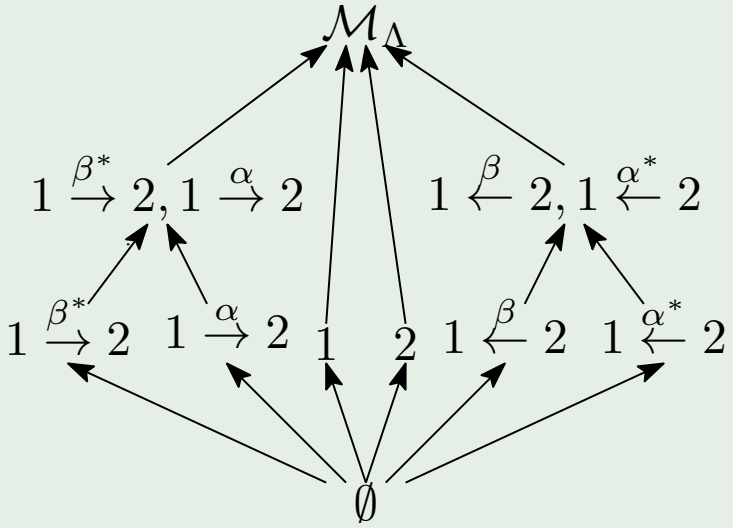
If Λ is a gentle algebra with no ouroboros, then there is an isomorphism $\Psi(\mathit{torshad}(\Lambda)) \cong \{\mathcal{W} \cap \mathcal{M}_\Lambda : \mathcal{W} \in \mathit{wide}(\Pi(\Lambda))\}$.

Example

$$\Lambda = \Bbbk\Big(\ 1 \underset{\beta}{\overset{\alpha}{\rightleftarrows}} 2\ \Big)/\langle \alpha\beta, \beta\alpha \rangle$$



torshad(Λ)



$\Psi(\mathrm{torshad}(\Lambda))$

Theorem (G.–McConville–Mousavand)

If Λ is a gentle algebra with no ouroboros, then there is an isomorphism $\Psi(torshad(\Lambda)) \cong \{\mathcal{W} \cap \mathcal{M}_\Lambda : \mathcal{W} \in wide(\Pi(\Lambda))\}$. We refer to the categories $\mathcal{W} \cap \mathcal{M}_\Lambda$ as ***wide shadows*** *and denote the lattice of wide shadows by $widshad(\Lambda)$.*

Idea of the proof

- For any $M(\widetilde{w}) \in \mathcal{M}_\Lambda$, $M(\widetilde{w})$ is a brick.
- For any distinct $M(\widetilde{w}_1), M(\widetilde{w}_2) \in \lambda_\downarrow(\mathcal{T} \cap \mathcal{M}_\Lambda)$ where $\mathcal{T} \cap \mathcal{M}_\Lambda \in \mathrm{torshad}(\Lambda)$, one has
$$\mathrm{Hom}_{\Pi(\Lambda)}(M(\widetilde{w}_i), M(\widetilde{w}_j)) = 0.$$
- By a theorem of Ringel, the category
$$filt(\lambda_\downarrow(\mathcal{T} \cap \mathcal{M}_\Lambda))$$
consisting of modules X with a filtration $0 = X_0 \subset X_1 \subset \cdots \subset X_k = X$ such that $X_i/X_{i-1} \in \lambda_\downarrow(\mathcal{T} \cap \mathcal{M}_\Lambda)$ is wide.
- Show that
$$\mathrm{add}(\oplus M(\widetilde{w}) : M(\widetilde{w}) \in \psi(\mathcal{T} \cap \mathcal{M}_\Lambda)) = filt(\lambda_\downarrow(\mathcal{T} \cap \mathcal{M}_\Lambda)) \cap \mathcal{M}_\Lambda.$$

Applications

Theorem (Marks–Šťovíček, 2015)

When Λ is an algebra of finite representation type, there is a bijection between torsion classes and wide subcategories given by

$$\begin{array}{ccc} tors(\Lambda) & \longrightarrow & wide(\Lambda) \\ \mathcal{T} & \longmapsto & \{X \in \mathcal{T} : (g : Y \to X) \in \mathcal{T}, ker(g) \in \mathcal{T}\} \\ filt(gen(\mathcal{W})) & \longleftarrow\!\shortmid & \mathcal{W}. \end{array}$$

Corollary

There is a bijection from torsion shadows to wide shadows given by

$$\begin{array}{ccc} torshad(\Lambda) & \longrightarrow & widshad(\Lambda) \\ \mathcal{T} \cap \mathcal{M}_\Lambda & \longmapsto & add(\oplus M(\widetilde{w}) : M(\widetilde{w}) \in \psi(\mathcal{T} \cap \mathcal{M}_\Lambda)) \\ filt(gen(\mathcal{W})) \cap \mathcal{M}_\Lambda & \longleftarrow\!\shortmid & \mathcal{W} \cap \mathcal{M}_\Lambda. \end{array}$$

Corollary

The poset widshad(Λ) is a lattice.

Proof.

- The category $\mathcal{M}_\Lambda$ is the unique maximal element.
- The poset widshad(Λ) is closed under intersections $(\mathcal{W}_1 \cap \mathcal{M}_\Lambda) \cap (\mathcal{W}_2 \cap \mathcal{M}_\Lambda) = (\mathcal{W}_1 \cap \mathcal{W}_2) \cap \mathcal{M}_\Lambda$.
- The poset widshad(Λ) is finite.

$\square$

Problem (Reading, 2016)

For which class of finite lattices L is $\Psi(L)$ a lattice? (If L is congruence-uniform, then $\Psi(L)$ is a partial order.)

Thanks!