

# Auslander's formula and the MacPherson-Vilonen Construction

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# Finitely presented functors

Throughout this talk,  $\mathcal{A}$  denotes an abelian category with enough projectives.

A functor  $F : \mathcal{A}^{\text{op}} \rightarrow \text{Ab}$  is **finitely presented** (or **coherent**) if there is a morphism  $f : A \rightarrow B \in \mathcal{A}$  and an exact sequence

$$\text{Hom}_{\mathcal{A}}(-, A) \xrightarrow{f_*} \text{Hom}_{\mathcal{A}}(-, B) \longrightarrow F \longrightarrow 0.$$

We write  $\text{fp}(\mathcal{A}^{\text{op}}, \text{Ab})$  for the category of all finitely presented functors  $\mathcal{A}^{\text{op}} \rightarrow \text{Ab}$  and natural transformations between them.

**Theorem (Auslander, 1965)**

*$\text{fp}(\mathcal{A}^{\text{op}}, \text{Ab})$  is an abelian category with global dimension 0 or 2.*

# The exact left adjoint of the Yoneda embedding

## Theorem (Auslander)

*The Yoneda embedding  $Y : \mathcal{A} \rightarrow \mathbf{fp}(\mathcal{A}^{\mathrm{op}}, \mathbf{Ab})$  has an exact left adjoint*

$$w : \mathbf{fp}(\mathcal{A}^{\mathrm{op}}, \mathbf{Ab}) \rightarrow \mathcal{A}.$$

*That is, for any  $F \in \mathbf{fp}(\mathcal{A}^{\mathrm{op}}, \mathbf{Ab})$  and  $A \in \mathcal{A}$ , there is an isomorphism*

$$\mathrm{Hom}_{\mathcal{A}}(wF, A) \cong \mathrm{Hom}_{\mathbf{fp}(\mathcal{A}^{\mathrm{op}}, \mathbf{Ab})}(F, \mathrm{Hom}_{\mathcal{A}}(-, A))$$

*which is natural in  $F$  and  $A$ .*

The counit of the adjunction  $w \dashv Y$  is an isomorphism  $wY \cong 1$ .

The unit of adjunction is the canonical map

$$1_{\mathbf{fp}(\mathcal{A}^{\mathrm{op}}, \mathbf{Ab})} \rightarrow R^0 \cong Yw.$$

# Auslander's formula

Since the functor  $w : \text{fp}(\mathcal{A}^{\text{op}}, \text{Ab}) \rightarrow \mathcal{A}$  is exact and has a fully faithful right adjoint  $Y : \mathcal{A} \rightarrow \text{fp}(\mathcal{A}^{\text{op}}, \text{Ab})$ . Therefore, it is a localisation, and in particular it is a Serre quotient. Therefore, we obtain **Auslander's formula**

$$\frac{\text{fp}(\mathcal{A}^{\text{op}}, \text{Ab})}{\text{fp}_0(\mathcal{A}^{\text{op}}, \text{Ab})} \simeq \mathcal{A},$$

where  $\text{fp}_0(\mathcal{A}^{\text{op}}, \text{Ab}) = \text{Ker}(w)$ .

# Describing $\mathrm{fp}_0(\mathcal{A}^{\mathrm{op}}, \mathrm{Ab})$

## Theorem (Auslander)

*For any  $F \in \mathrm{fp}(\mathcal{A}^{\mathrm{op}}, \mathrm{Ab})$ , the following are equivalent.*

1.  $F \in \mathrm{fp}_0(\mathcal{A}^{\mathrm{op}}, \mathrm{Ab})$ , i.e.  $wF = 0$ .
2. *For any projective presentation*

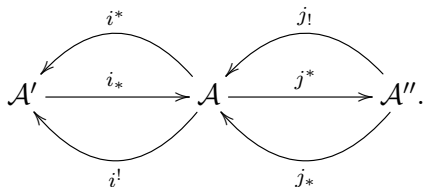
$$\mathrm{Hom}_{\mathcal{A}}(-, A) \xrightarrow{f_*} \mathrm{Hom}_{\mathcal{A}}(-, B) \longrightarrow F \longrightarrow 0.$$

*the morphism  $f : A \rightarrow B$  is an epimorphism*

3.  $\mathrm{Hom}_{\mathrm{fp}(\mathcal{A}^{\mathrm{op}}, \mathrm{Ab})}(F, \mathrm{Hom}_{\mathcal{A}}(-, X)) = 0$  for any  $X \in \mathcal{A}$ .

# Abelian recollements

A **recollement** (of abelian categories) is a situation



in which  $\mathcal{A}'$ ,  $\mathcal{A}$  and  $\mathcal{A}''$  are abelian categories and the following hold:

- ▶  $i^* \dashv i_* \dashv i^!$
- ▶  $j_* \dashv j^* \dashv j^!$
- ▶  $i_*$ ,  $j^!$  and  $j_*$  are fully faithful.
- ▶  $\text{Im}(i_*) = \text{Ker}(j^*)$ .

# Auslander's formula – recollement form

Theorem (SD, Jeremy Russell)

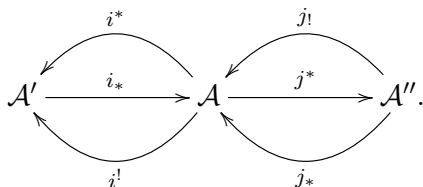
*There is a recollement*

$$\begin{array}{ccccc} & & (-)^0 & & L_0(Y) \\ & \swarrow & & \searrow & \\ \mathrm{fp}_0(\mathcal{A}^{\mathrm{op}}, \mathrm{Ab}) & \xrightarrow{\subseteq} & \mathrm{fp}(\mathcal{A}^{\mathrm{op}}, \mathrm{Ab}) & \xrightarrow{w} & \mathcal{A} \\ & \nwarrow & & \nearrow & \\ & & (-)_0 & & Y \end{array}$$

- ▶  $L_0(Y)(P) = \mathrm{Hom}_{\mathcal{A}}(-, P)$  for any projective  $P \in \mathcal{A}$ .
- ▶  $(\mathrm{Hom}_{\mathcal{A}}(-, A))^0 = \underline{\mathrm{Hom}}_{\mathcal{A}}(-, A)$  for any  $A \in \mathcal{A}$ .
- ▶  $F_0 A = (\underline{\mathrm{Hom}}_{\mathcal{A}}(-, A), F)$  for any  $F \in \mathrm{fp}(\mathcal{A}^{\mathrm{op}}, \mathrm{Ab})$  and  $A \in \mathcal{A}$ .

# Pre-hereditary recollements

A recollement



is said to be **pre-hereditary** if  $L_2(i^*)(i_*V) = 0$  for each projective object  $V \in \mathcal{A}'$ .

**Who cares?** If  $\mathcal{B}'$ , and  $\mathcal{B}''$  are triangulated categories and  $\mathcal{B}_1$  and  $\mathcal{B}_2$  are recollements of  $\mathcal{B}'$  and  $\mathcal{B}''$ , then any functor  $\mathcal{B}_1 \rightarrow \mathcal{B}_2$  which respects all of the recollement structures is a triangulated equivalence. This doesn't hold for recollements of abelian categories, but it does hold for pre-hereditary recollements.



# When is our recollement pre-hereditary?

In the recollement

$$\begin{array}{ccccc}
 & & (-)^0 & & L_0(Y) \\
 & \swarrow & & \searrow & \\
 \mathrm{fp}_0(\mathcal{A}^{\mathrm{op}}, \mathrm{Ab}) & \xrightarrow{\subseteq} & \mathrm{fp}(\mathcal{A}^{\mathrm{op}}, \mathrm{Ab}) & \xrightarrow{w} & \mathcal{A}. \\
 & \nwarrow & & \nearrow & \\
 & & (-)_0 & & Y
 \end{array}$$

$i^* = (-)^0$  so it is pre-hereditary if and only if

$$\mathrm{L}_2((-)^0)(V) = 0$$

for every projective  $V = \underline{\mathrm{Hom}}_{\mathcal{A}}(-, A)$  of  $\mathrm{fp}_0(\mathcal{A}^{\mathrm{op}}, \mathrm{Ab})$ .

# When is our recollement pre-hereditary?

## Lemma

$L_2((-)^0)(\underline{\mathrm{Hom}}_{\mathcal{A}}(-, A)) = \underline{\mathrm{Hom}}_{\mathcal{A}}(-, \Omega A)$  for every  $A \in \mathcal{A}$ .

## Corollary

*The recollement*

$$\begin{array}{ccccc} & & (-)^0 & & L_0(Y) \\ & \swarrow & & \searrow & \\ \mathrm{fp}_0(\mathcal{A}^{\mathrm{op}}, \mathrm{Ab}) & \xrightarrow{\subseteq} & \mathrm{fp}(\mathcal{A}^{\mathrm{op}}, \mathrm{Ab}) & \xrightarrow{w} & \mathcal{A} \\ & \nwarrow & & \nearrow & \\ & & (-)_0 & & Y \end{array}$$

is pre-hereditary if and only if  $\mathcal{A}$  has global dimension at most one (i.e.  $\mathcal{A}$  is a **hereditary abelian category**).

# The MacPherson-Vilonen construction

Let  $\mathcal{A}'$  and  $\mathcal{A}''$  be abelian categories, let  $F : \mathcal{A}'' \rightarrow \mathcal{A}'$  be a right exact functor, let  $G : \mathcal{A}'' \rightarrow \mathcal{A}'$  be a left exact functor, and let  $\alpha : F \rightarrow G$  be a natural transformation. The **MacPherson-Vilonen construction** for  $\alpha$  is recollement of abelian categories

$$\begin{array}{ccccc} & i^* & & j^! & \\ & \curvearrowleft & & \curvearrowleft & \\ \mathcal{A}' & \xrightarrow{i_*} & \mathcal{A}(\alpha) & \xrightarrow{j^*} & \mathcal{A}'' \\ & \curvearrowright & & \curvearrowright & \\ & i_! & & j_* & \end{array}$$

given by the following data...

# The MacPherson-Vilonen construction

- Objects  $(X, V, g, f)$  given by an object  $X \in \mathcal{A}''$ , an object  $V \in \mathcal{A}'$  and morphisms

$$FX \xrightarrow{f} V \xrightarrow{g} GX$$

such that  $gf = \alpha_X$ .

- Morphisms  $(x, v) : (X, V, g, f) \rightarrow (X', V', g', f')$  given by morphisms  $x : X \rightarrow X' \in \mathcal{A}'$  and  $v : V \rightarrow V' \in \mathcal{A}''$  such that the diagram

$$\begin{array}{ccccc} FX & \xrightarrow{f} & V & \xrightarrow{g} & GX \\ \downarrow Fx & & \downarrow v & & \downarrow Gx \\ FX' & \xrightarrow{f'} & V' & \xrightarrow{g'} & GX' \end{array}$$

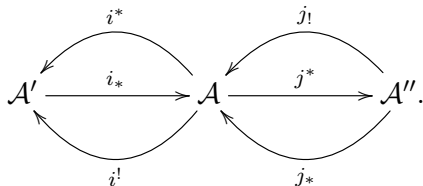
commutes.

# When is our recollement MP-V?

We will apply the following result.

Theorem (Franjou, Pirashvili)

*A recollement*



*is an instance of the MacPherson-Vilonen construction if and only if the following hold:*

1. *It is pre-hereditary.*
2. *There is an exact functor  $p : \mathcal{A} \rightarrow \mathcal{A}'$  such that  $pi_* \cong 1_{\mathcal{A}'}$ .*

Answer: Pre-hereditary implies MP-V for our recollement

### Lemma

If  $\mathcal{A}$  is hereditary then the functor  $\mathcal{A} \rightarrow \text{fp}_0(\mathcal{A}^{\text{op}}, \text{Ab}) : A \mapsto \underline{\text{Hom}}_{\mathcal{A}}(-, A)$  is left exact.

### Proof.

There is an equivalence  $W : (\text{fp}_0(\mathcal{A}, \text{Ab}))^{\text{op}} \rightarrow \text{fp}(\mathcal{A}^{\text{op}}, \text{Ab})$  such that  $W\text{Ext}^1(A, -) = \underline{\text{Hom}}_{\mathcal{A}}(-, A)$  for all  $A \in \mathcal{A}$ . Since  $A \mapsto \text{Ext}^1(A, -)$  is right exact, this is enough.  $\square$

### Lemma

If  $\mathcal{A}$  is hereditary then the functor  $(-)^0 : \text{fp}(\mathcal{A}^{\text{op}}, \text{Ab}) \rightarrow \text{fp}_0(\mathcal{A}^{\text{op}}, \text{Ab})$  is exact.

### Proof.

Using above result, one can show that  $L_1((-)^0) = 0$ .  $\square$

# The final result

$$\begin{array}{ccccc}
 & & (-)^0 & & L_0(Y) \\
 & \swarrow & & \searrow & \\
 \mathrm{fp}_0(\mathcal{A}^{\mathrm{op}}, \mathrm{Ab}) & \xrightarrow{\subseteq} & \mathrm{fp}(\mathcal{A}^{\mathrm{op}}, \mathrm{Ab}) & \xrightarrow{w} & \mathcal{A}. \\
 & \nwarrow & & \nearrow & \\
 & & (-)_0 & & Y
 \end{array}$$

## Theorem

*The following are equivalent for the above recollement.*

1. *The recollement is pre-hereditary.*
2. *The recollement is MacPherson-Vilonen.*
3. *The category  $\mathcal{A}$  is hereditary.*

*If it is MacPherson-Vilonen, then it is the MacPherson-Vilonen construction for  $0 \rightarrow \underline{Y}$ , where  $\underline{Y} : \mathcal{A} \rightarrow \mathrm{fp}_0(\mathcal{A}^{\mathrm{op}}, \mathrm{Ab})$  is given by  $\underline{Y}A = \underline{\mathrm{Hom}}_{\mathcal{A}}(-, A)$  for  $A \in \mathcal{A}$ .*

# A Serre quotient formula for hereditary categories\*

During this talk, we showed that if  $\mathcal{A}$  is hereditary then the functor

$$(-)^0 : \mathrm{fp}(\mathcal{A}^{\mathrm{op}}, \mathrm{Ab}) \rightarrow \mathrm{fp}_0(\mathcal{A}^{\mathrm{op}}, \mathrm{Ab})$$

is exact. It also has a fully faithful right adjoint – the embedding  $\mathrm{fp}_0(\mathcal{A}^{\mathrm{op}}, \mathrm{Ab}) \hookrightarrow \mathrm{fp}(\mathcal{A}^{\mathrm{op}}, \mathrm{Ab})$ . Therefore, if  $\mathcal{A}$  is hereditary,  $(-)^0$  is a localisation, hence a Serre quotient, and we obtain an equivalence

$$\frac{\mathrm{fp}(\mathcal{A}^{\mathrm{op}}, \mathrm{Ab})}{\mathrm{fp}_1(\mathcal{A}^{\mathrm{op}}, \mathrm{Ab})} \simeq \mathrm{fp}_0(\mathcal{A}^{\mathrm{op}}, \mathrm{Ab}),$$

where

$$\mathrm{fp}_1(\mathcal{A}^{\mathrm{op}}, \mathrm{Ab}) = \mathrm{Ker}((-)^0).$$



# A description of $\text{fp}_1(\mathcal{A}^{\text{op}}, \text{Ab})$

Now we drop the assumption that  $\mathcal{A}$  is hereditary.

## Theorem

*For any functor  $F \in \text{fp}(\mathcal{A}^{\text{op}}, \text{Ab})$ , the following are equivalent.*

1.  $F \in \text{fp}_1(\mathcal{A}^{\text{op}}, \text{Ab})$ , i.e.  $F^0 = 0$ .
2. *For any projective presentation*

$$\text{Hom}_{\mathcal{A}}(-, A) \xrightarrow{f_*} \text{Hom}_{\mathcal{A}}(-, B) \twoheadrightarrow F \twoheadrightarrow 0$$

*the map  $\underline{f} : \underline{A} \rightarrow \underline{B}$  is a split epimorphism in  $\underline{\mathcal{A}}$ .*

3.  $\text{Hom}_{\text{fp}(\mathcal{A}^{\text{op}}, \text{Ab})}(F, \text{Ext}_{\mathcal{A}}^1(-, A)) = 0$  for any  $A \in \mathcal{A}$ .

## Theorem

*If  $\mathcal{A}$  has enough injectives then  $(\text{fp}_1(\mathcal{A}^{\text{op}}, \text{Ab}), \text{fp}_0(\mathcal{A}^{\text{op}}, \text{Ab}))$  is a torsion theory in  $\text{fp}(\mathcal{A}^{\text{op}}, \text{Ab})$ .*