Auslander's formula and the MacPherson-Vilonen Construction

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Finitely presented functors

Throughout this talk, $\mathcal{A}$ denotes an abelian category with enough projectives.

A functor $F : \mathcal{A}^{\mathrm{op}} \to \mathrm{Ab}$ is **finitely presented** (or **coherent**) if there is a morphism $f : A \to B \in \mathcal{A}$ and an exact sequence

$$\mathrm{Hom}_{\mathcal{A}}(-, A) \xrightarrow{f_*} \mathrm{Hom}_{\mathcal{A}}(-, B) \longrightarrow F \longrightarrow 0.$$

We write $\mathrm{fp}(\mathcal{A}^{\mathrm{op}}, \mathrm{Ab})$ for the category of all finitely presented functors $\mathcal{A}^{\mathrm{op}} \to \mathrm{Ab}$ and natural transformations between them.

Theorem (Auslander, 1965)

$\mathrm{fp}(\mathcal{A}^{\mathrm{op}}, \mathrm{Ab})$ *is an abelian category with global dimension* 0 *or* 2.

The exact left adjoint of the Yoneda embedding

Theorem (Auslander)

The Yoneda embedding $Y : \mathcal{A} \to \mathrm{fp}(\mathcal{A}^{\mathrm{op}}, \mathrm{Ab})$ *has an exact left adjoint*

$$w : \mathrm{fp}(\mathcal{A}^{\mathrm{op}}, \mathrm{Ab}) \to \mathcal{A}.$$

That is, for any $F \in \mathrm{fp}(\mathcal{A}^{\mathrm{op}}, \mathrm{Ab})$ *and* $A \in \mathcal{A}$*, there is an isomorphism*

$$\mathrm{Hom}_{\mathcal{A}}(wF, A) \cong \mathrm{Hom}_{\mathrm{fp}(\mathcal{A}^{\mathrm{op}}, \mathrm{Ab})}(F, \mathrm{Hom}_{\mathcal{A}}(-, A))$$

which is natural in F *and* A*.*

The counit of the adjunction $w \dashv Y$ is an isomorphism $wY \cong 1$. The unit of adjunction is the canonical map

$$1_{\mathrm{fp}(\mathcal{A}^{\mathrm{op}}, \mathrm{Ab})} \to R^0 \cong Yw.$$

Auslander's formula

Since the functor $w : \mathrm{fp}(\mathcal{A}^{\mathrm{op}}, \mathrm{Ab}) \to \mathcal{A}$ is exact and has a fully faithful right adjoint $Y : \mathcal{A} \to \mathrm{fp}(\mathcal{A}^{\mathrm{op}}, \mathrm{Ab})$. Therefore, it is a localisation, and in particular it is a Serre quotient. Therefore, we obtain **Auslander's formula**

$$\frac{\mathrm{fp}(\mathcal{A}^{\mathrm{op}}, \mathrm{Ab})}{\mathrm{fp}_0(\mathcal{A}^{\mathrm{op}}, \mathrm{Ab})} \simeq \mathcal{A},$$

where $\mathrm{fp}_0(\mathcal{A}^{\mathrm{op}}, \mathrm{Ab}) = \mathrm{Ker}(w)$.

Describing $\mathrm{fp}_0(\mathcal{A}^{\mathrm{op}}, \mathrm{Ab})$

Theorem (Auslander)

For any $F \in \mathrm{fp}(\mathcal{A}^{\mathrm{op}}, \mathrm{Ab})$, the following are equivalent.

1. $F \in \mathrm{fp}_0(\mathcal{A}^{\mathrm{op}}, \mathrm{Ab})$, *i.e.* $wF = 0$.
2. *For any projective presentation*

$$\mathrm{Hom}_{\mathcal{A}}(-, A) \xrightarrow{f_*} \mathrm{Hom}_{\mathcal{A}}(-, B) \longrightarrow F \longrightarrow 0.$$

   the morphism $f : A \to B$ is an epimorphism
3. $\mathrm{Hom}_{\mathrm{fp}(\mathcal{A}^{\mathrm{op}},\mathrm{Ab})}(F, \mathrm{Hom}_{\mathcal{A}}(-, X)) = 0$ *for any $X \in \mathcal{A}$.*

Abelian recollements

A **recollement (of abelian categories)** is a situation

$$\begin{tikzcd} \mathcal{A}' \arrow[r, "i_*"] & \mathcal{A} \arrow[l, bend right=60, "i^*"'] \arrow[l, bend left=60, "i^!"] \arrow[r, "j^*"] & \mathcal{A}''. \arrow[l, bend right=60, "j_!"'] \arrow[l, bend left=60, "j_*"] \end{tikzcd}$$

in which $\mathcal{A}'$, $\mathcal{A}$ and $\mathcal{A}''$ are abelian categories and the following hold:

- $i^* \dashv i_* \dashv i^!$
- $j_* \dashv j^* \dashv j_!$
- i_*, $j_!$ and j_* are fully faithful.
- $\mathrm{Im}(i_*) = \mathrm{Ker}(j^*)$.

Auslander's formula – recollement form

Theorem (SD, Jeremy Russell)

There is a recollement

$$\begin{tikzcd} \mathrm{fp}_0(\mathcal{A}^{\mathrm{op}}, \mathrm{Ab}) \arrow[r, "\subseteq"] & \mathrm{fp}(\mathcal{A}^{\mathrm{op}}, \mathrm{Ab}) \arrow[l, bend right, "(-)^0"'] \arrow[l, bend left, "(-)_0"] \arrow[r, "w"] & \mathcal{A}. \arrow[l, bend right, "L_0(Y)"'] \arrow[l, bend left, "Y"] \end{tikzcd}$$

- $L_0(Y)(P) = \mathrm{Hom}_{\mathcal{A}}(-, P)$ *for any projective* $P \in \mathcal{A}$.
- $(\mathrm{Hom}_{\mathcal{A}}(-, A))^0 = \underline{\mathrm{Hom}}_{\mathcal{A}}(-, A)$ *for any* $A \in \mathcal{A}$.
- $F_0 A = (\underline{\mathrm{Hom}}_{\mathcal{A}}(-, A), F)$ *for any* $F \in \mathrm{fp}(\mathcal{A}^{\mathrm{op}}, \mathrm{Ab})$ *and* $A \in \mathcal{A}$.

Pre-hereditary recollements

A recollement

$$\begin{tikzcd}
\mathcal{A}' \arrow[r, "i_*"] & \mathcal{A} \arrow[l, bend right, "i^*"'] \arrow[l, bend left, "i^!"] \arrow[r, "j^*"] & \mathcal{A}''. \arrow[l, bend right, "j_!"'] \arrow[l, bend left, "j_*"]
\end{tikzcd}$$

is said to be **pre-hereditary** if $\mathrm{L}_2(i^*)(i_*V) = 0$ for each projective object $V \in \mathcal{A}'$.

Who cares? If $\mathcal{B}'$, and $\mathcal{B}''$ are triangulated categories and $\mathcal{B}_1$ and $\mathcal{B}_2$ are recollements of $\mathcal{B}'$ and $\mathcal{B}''$, then any functor $\mathcal{B}_1 \to \mathcal{B}_2$ which respects all of the recollement structures is a triangulated equivalence. This doesn't hold for recollements of abelian categories, but it does hold for pre-hereditary recollements.

When is our recollement pre-hereditary?

In the recollement

$$\begin{tikzcd}[column sep=huge]
\mathrm{fp}_0(\mathcal{A}^{\mathrm{op}}, \mathrm{Ab}) \arrow[r, "\subseteq"] & \mathrm{fp}(\mathcal{A}^{\mathrm{op}}, \mathrm{Ab}) \arrow[l, bend right, "(-)^0"'] \arrow[l, bend left, "(-)_0"] \arrow[r, "w"] & \mathcal{A}. \arrow[l, bend right, "L_0(Y)"'] \arrow[l, bend left, "Y"]
\end{tikzcd}$$

$i^* = (-)^0$ so it is pre-hereditary if and only if

$$\mathrm{L}_2((-)^0)(V) = 0$$

for every projective $V = \underline{\mathrm{Hom}}_{\mathcal{A}}(-, A)$ of $\mathrm{fp}_0(\mathcal{A}^{\mathrm{op}}, \mathrm{Ab})$.

When is our recollement pre-hereditary?

Lemma

$\mathrm{L}_2((-)^0)(\underline{\mathrm{Hom}}_{\mathcal{A}}(-,A)) = \underline{\mathrm{Hom}}_{\mathcal{A}}(-,\Omega A)$ *for every* $A \in \mathcal{A}$*.*

Corollary

The recollement

$$\begin{tikzcd} \mathrm{fp}_0(\mathcal{A}^{\mathrm{op}},\mathrm{Ab}) \arrow[rr, "\subseteq"] & & \mathrm{fp}(\mathcal{A}^{\mathrm{op}},\mathrm{Ab}) \arrow[ll, bend right, "(-)^0"'] \arrow[ll, bend left, "(-)_0"] \arrow[rr, "w"] & & \mathcal{A}. \arrow[ll, bend right, "L_0(Y)"'] \arrow[ll, bend left, "Y"] \end{tikzcd}$$

is pre-hereditary if and only if $\mathcal{A}$ *has global dimension at most one (i.e.* $\mathcal{A}$ *is a* ***hereditary abelian category****).*

The MacPherson-Vilonen construction

Let $\mathcal{A}'$ and $\mathcal{A}''$ be abelian categories, let $F : \mathcal{A}'' \to \mathcal{A}'$ be a right exact functor, let $G : \mathcal{A}'' \to \mathcal{A}'$ be a left exact functor, and let $\alpha : F \to G$ be a natural transformation. The **MacPherson-Vilonen construction** for α is recollement of abelian categories

$$\begin{tikzcd} \mathcal{A}' \arrow[r, "i_*"] & \mathcal{A}(\alpha) \arrow[l, bend right, "i^*"'] \arrow[l, bend left, "i^!"] \arrow[r, "j^*"] & \mathcal{A}''. \arrow[l, bend right, "j_!"'] \arrow[l, bend left, "j_*"] \end{tikzcd}$$

given by the following data...

The MacPherson-Vilonen construction

- Objects (X, V, g, f) given by an object $X \in \mathcal{A}''$, an object $V \in \mathcal{A}'$ and morphisms

$$FX \xrightarrow{f} V \xrightarrow{g} GX$$

such that $gf = \alpha_X$.

- Morphisms $(x, v) : (X, V, g, f) \to (X', V', g', f')$ given by morphisms $x : X \to X' \in \mathcal{A}'$ and $v : V \to V' \in \mathcal{A}''$ such that the diagram

$$\begin{tikzcd}
FX \arrow[r, "f"] \arrow[d, "Fx"'] & V \arrow[r, "g"] \arrow[d, "v"] & GX \arrow[d, "Gx"] \\
FX' \arrow[r, "f'"] & V' \arrow[r, "g'"] & GX'
\end{tikzcd}$$

commutes.

When is our recollement MP-V?

We will apply the following result.

Theorem (Franjou, Pirashvili)

A recolleement

$$\begin{tikzcd} \mathcal{A}' \arrow[r, "i_*"] & \mathcal{A} \arrow[l, bend right, "i^*"'] \arrow[l, bend left, "i^!"] \arrow[r, "j^*"] & \mathcal{A}''. \arrow[l, bend right, "j_!"'] \arrow[l, bend left, "j_*"] \end{tikzcd}$$

is an instance of the MacPherson-Vilonen construction if and only if the following hold:

1. *It is pre-hereditary.*
2. *There is an exact functor* $p : \mathcal{A} \to \mathcal{A}'$ *such that* $pi_* \cong 1_{\mathcal{A}'}$.

Answer: Pre-hereditary implies MP-V for our recollement

Lemma

If $\mathcal{A}$ is hereditary then the functor
$\mathcal{A} \to \mathrm{fp}_0(\mathcal{A}^{\mathrm{op}}, \mathrm{Ab}) : A \mapsto \underline{\mathrm{Hom}}_{\mathcal{A}}(-, A)$ *is left exact.*

Proof.

There is an equivalence $W : (\mathrm{fp}_0(\mathcal{A}, \mathrm{Ab}))^{\mathrm{op}} \to \mathrm{fp}(\mathcal{A}^{\mathrm{op}}, \mathrm{Ab})$ such that $W\mathrm{Ext}^1(A, -) = \underline{\mathrm{Hom}}_{\mathcal{A}}(-, A)$ for all $A \in \mathcal{A}$. Since $A \mapsto \mathrm{Ext}^1(A, -)$ is right exact, this is enough. □

Lemma

If $\mathcal{A}$ is hereditary then the functor
$(-)^0 : \mathrm{fp}(\mathcal{A}^{\mathrm{op}}, \mathrm{Ab}) \to \mathrm{fp}_0(\mathcal{A}^{\mathrm{op}}, \mathrm{Ab})$ *is exact.*

Proof.

Using above result, one can show that $\mathrm{L}_1((-)^0) = 0$. □

The final result

$$\begin{tikzcd}[column sep=huge]
\mathrm{fp}_0(\mathcal{A}^{\mathrm{op}}, \mathrm{Ab}) \arrow[r, "\subseteq"] & \mathrm{fp}(\mathcal{A}^{\mathrm{op}}, \mathrm{Ab}) \arrow[l, bend right, "(-)^0"'] \arrow[l, bend left, "(-)_0"] \arrow[r, "w"] & \mathcal{A}. \arrow[l, bend right, "L_0(Y)"'] \arrow[l, bend left, "Y"]
\end{tikzcd}$$

Theorem

The following are equivalent for the above recollement.

1. *The recollement is pre-hereditary.*
2. *The recollement is MacPherson-Vilonen.*
3. *The category $\mathcal{A}$ is hereditary.*

If it is MacPherson-Vilonen, then it is the MacPherson-Vilonen construction for $0 \to \underline{Y}$, where $\underline{Y} : \mathcal{A} \to \mathrm{fp}_0(\mathcal{A}^{\mathrm{op}}, \mathrm{Ab})$ is given by $\underline{Y}A = \underline{\mathrm{Hom}}_{\mathcal{A}}(-, A)$ for $A \in \mathcal{A}$.

A Serre quotient formula for hereditary categories*

During this talk, we showed that if $\mathcal{A}$ is hereditary then the functor

$$(-)^0 : \mathrm{fp}(\mathcal{A}^{\mathrm{op}}, \mathrm{Ab}) \to \mathrm{fp}_0(\mathcal{A}^{\mathrm{op}}, \mathrm{Ab})$$

is exact. It also has a fully faithful right adjoint – the embedding $\mathrm{fp}_0(\mathcal{A}^{\mathrm{op}}, \mathrm{Ab}) \hookrightarrow \mathrm{fp}(\mathcal{A}^{\mathrm{op}}, \mathrm{Ab})$. Therefore, if $\mathcal{A}$ is hereditary, $(-)^0$ is a localisation, hence a Serre quotient, and we obtain an equivalence

$$\frac{\mathrm{fp}(\mathcal{A}^{\mathrm{op}}, \mathrm{Ab})}{\mathrm{fp}_1(\mathcal{A}^{\mathrm{op}}, \mathrm{Ab})} \simeq \mathrm{fp}_0(\mathcal{A}^{\mathrm{op}}, \mathrm{Ab}),$$

where

$$\mathrm{fp}_1(\mathcal{A}^{\mathrm{op}}, \mathrm{Ab}) = \mathrm{Ker}((-)^0).$$

A description of $\mathrm{fp}_1(\mathcal{A}^{\mathrm{op}}, \mathrm{Ab})$

Now we drop the assumption that $\mathcal{A}$ is hereditary.

Theorem

For any functor $F \in \mathrm{fp}(\mathcal{A}^{\mathrm{op}}, \mathrm{Ab})$, the following are equivalent.

1. $F \in \mathrm{fp}_1(\mathcal{A}^{\mathrm{op}}, \mathrm{Ab})$, *i.e.* $F^0 = 0$.
2. *For any projective presentation*

$$\mathrm{Hom}_{\mathcal{A}}(-, A) \xrightarrow{f_*} \mathrm{Hom}_{\mathcal{A}}(-, B) \longrightarrow F \longrightarrow 0$$

   the map $\underline{f} : \underline{A} \to \underline{B}$ *is a split epimorphism in* $\underline{\mathcal{A}}$.
3. $\mathrm{Hom}_{\mathrm{fp}(\mathcal{A}^{\mathrm{op}}, \mathrm{Ab})}(F, \mathrm{Ext}^1_{\mathcal{A}}(-, A)) = 0$ *for any* $A \in \mathcal{A}$.

Theorem

If $\mathcal{A}$ has enough injectives then $(\mathrm{fp}_1(\mathcal{A}^{\mathrm{op}}, \mathrm{Ab}), \mathrm{fp}_0(\mathcal{A}^{\mathrm{op}}, \mathrm{Ab}))$ *is a torsion theory in* $\mathrm{fp}(\mathcal{A}^{\mathrm{op}}, \mathrm{Ab})$.