Top exterior powers in Iwasawa theory

Frauke Bleher

joint with Ted Chinburg, Ralph Greenberg, Mahesh Kakde,
Romyar Sharifi and Martin Taylor

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Iwasawa theory (started by Kenkichi Iwasawa 1950's).

- Study growth rates of numerical invariants associated to infinite towers of number fields.
  - Classical numerical invariants: logarithms of the sizes of the p-parts of ideal class groups as one moves up such a tower.
- Algebraic description: The pro-group ring over $\mathbb{Z}_p$ of the Galois group of an appropriate infinite tower of number fields is called an Iwasawa algebra. The growth rates of the numerical invariants of interest are determined from the structure of particular modules for such Iwasawa algebras. Such modules are called Iwasawa modules.
- Analytic invariants: Given by p-adic L-functions which are defined by p-adically interpolating the values at negative integers of L-functions associated to Dirichlet characters, Grössencharacters or modular forms.
- Main Conjectures of Iwasawa theory: The structure invariants of the appropriate Iwasawa modules are determined by p-adic L-functions.

More details.

Iwasawa theory produces from Galois theory certain finitely generated torsion modules M for a Noetherian UFD R.

In practice $R = W[[t_1, ..., t_r]]$ for some integer $r \geq 1$, where $W = W(\overline{\mathbb{F}}_p)$ is the ring of infinite Witt vectors over $\overline{\mathbb{F}}_p$.

The Main Conjectures of Iwasawa theory that have been studied up till now have to do with the first Chern class of M:

$$c_1(M) = \sum_{v \in V^{(1)}} \mathrm{length}_{R_v}(M_v) \cdot v \quad \in \quad \bigoplus_{v \in V^{(1)}} \mathbb{Z} \cdot v$$

where $V^{(1)}$ is the set of all codimension 1 (i.e., height 1) prime ideals in $V = \mathrm{Spec}(R)$.

Main Conjectures have to do with showing

$$c_1(M) = c_1(R/RL)$$

where L is a certain p-adic L-function in R.

Greenberg's conjecture.

A conjecture by Greenberg predicts that in various natural situations $c_1(M) = 0$, i.e. all the localizations of M at codimension one primes are trivial. Such an M is said to be pseudo-null.

Over the last several years, my collaborators and I have studied the natural second Chern class $c_2(M)$ in this case, which is defined as

$$c_2(M) = \sum_{v \in V^{(2)}} \mathrm{length}_{R_v}(M_v) \cdot v \quad \in \quad \bigoplus_{v \in V^{(2)}} \mathbb{Z} \cdot v$$

where $V^{(2)}$ is the set of all codimension 2 primes in $V = \mathrm{Spec}(R)$.

The natural hope is that $c_2(M)$ is related to

$$c_2\left(\frac{R}{RL_1 + RL_2}\right)$$

when L_1 and L_2 are two different p-adic L-functions in R arising from the Iwasawa theoretic data.

More detail on growth rates.

A reason for considering the higher Chern classes of M for which $c_1(M) = 0$ is to get control on the leading term in the growth rates of numerical invariants associated to towers of number fields.

Higher Chern classes are the analogs of the leading terms in the Taylor expansions of functions of a real variable. The first Chern classes are analogous to first derivatives. When the first derivative vanishes, the story is not over. The natural question then is to consider higher derivatives.

We are asking the analogous question in Iwasawa theory.

A strategy for studying Iwasawa modules.

I will describe later a particular Iwasawa module X for an Iwasawa algebra R arising from a tower of number fields over a CM number field E. Such E are quadratic extensions of totally real fields.

Define $X^* = \mathrm{Hom}_R(X, R)$. Our strategy is to consider the natural homomorphism

$$X \to X^{**} = (X^*)^*.$$

Its kernel is the R-torsion submodule $\mathrm{Tor}_R(X)$ of X.

We control the cokernel of this homomorphism via étale duality theorems of McCallum, Jannsen and Nekovář.

The advantage of X^{**} is that its structure is in general simpler than that of X. In particular, the localization of X^{**} at a codimension 1 or 2 prime of R will be free.

General set up: Suppose one has an R-module homomorphism

$$\lambda : X \to F$$

between finitely generated R-modules inducing an isomorphism when tensoring with $\mathrm{Frac}(R)$ and in which F is free of rank ℓ.

Iwasawa module context:

Localize at a codimension 1 or 2 prime, and let $F = X^{**}$.

Iwasawa theory also produces free rank ℓ submodules I_1, I_2 of X that map isomorphically to their images J_1, J_2 of F, with first Chern classes

$$c_1(X/I_1) = c_1(R/RL_1) \quad \text{and} \quad c_1(X/I_2) = c_1(R/RL_2).$$

OUR GOAL:

Use this information to arrive at an expression for $c_2(M')$ when M' is the maximal pseudo-null submodule of $R/(RL_1 + RL_2)$.

Number theoretic comment.

We are flipping the usual approach to Iwasawa theory by focusing first on the natural R-modules that can be constructed using p-adic L-functions.

- Namely, we consider the module M' that is the maximal pseudo-null submodule of $R/(RL_1 + RL_2)$.
- We then look for Galois theoretic modules M that will have second and first Chern classes related to $c_2(M')$.

This method is similar to what happens in Stark's Conjecture, which is about finding algebraic interpretations of the leading terms in the Taylor expansions of Artin L-functions. One main tool there is to use top exterior powers of isotypic components of unit groups of number fields.

We have found that in Iwasawa theory, one similarly can use the top exterior powers of Iwasawa modules to study Chern classes that are defined analytically using p-adic L-functions.

Lemma 1:

Let R be an integral domain. Suppose one has an R-module homomorphism

$$\lambda : X \to F$$

between finitely generated R-modules inducing an isomorphism when tensoring with $\mathrm{Frac}(R)$ and in which F is free of rank ℓ.
Define $Y = \mathrm{Coker}(\lambda)$.
Then we have an exact sequence of R-modules

$$0 \to \mathrm{Tor}_R(\wedge^\ell X) \to \wedge^\ell X \xrightarrow{\wedge^\ell \lambda} \wedge^\ell F \to \frac{R}{\mathrm{Fitt}_0(Y)} \to 0.$$

where $\mathrm{Tor}_R(\wedge^\ell X)$ is the R-torsion submodule of $\wedge^\ell X$, and $\mathrm{Fitt}_0(Y)$ is the 0^{th} Fitting ideal of Y.

Recall:

Suppose Y is generated as an R-module by the elements

$$y_1, \ldots, y_n$$

with relations

$$a_{j1}y_1 + a_{j2}y_2 + \cdots + a_{jn}y_n = 0 \qquad \text{for } j = 1, 2, \ldots$$

The 0^{th} Fitting ideal $\mathrm{Fitt}_0(Y)$ is the ideal of R generated by the $n \times n$ minors of the matrix (a_{jk}).

Note: $\mathrm{Fitt}_0(Y)$ does not depend on the choice of generators and relations of Y.

Let $F = R^\ell$ and $\lambda : X \to F$ be as before, where λ induces an isomorphism when tensored with $\mathrm{Frac}(R)$. Let $Y = \mathrm{Coker}(\lambda)$.

Lemma 1:

We have a short exact exact sequence of R-modules

$$0 \to \frac{\wedge^\ell X}{\mathrm{Tor}_R(\wedge^\ell X)} \xrightarrow{\overline{\wedge^\ell \lambda}} \wedge^\ell F \to \frac{R}{\mathrm{Fitt}_0(Y)} \to 0.$$

Lemma 2:

Let I_1, I_2 be free rank ℓ submodules of X, define $J_i = \lambda(I_i) \subset F$.
The image of $\wedge^\ell I_i$ in $\wedge^\ell X$ is isomorphic to $\wedge^\ell I_i \Rightarrow$ call it $\wedge^\ell I_i$.
The image of $\wedge^\ell J_i$ in $\wedge^\ell F$ is isomorphic to $\wedge^\ell J_i \Rightarrow$ call it $\wedge^\ell J_i$.
Moreover, $\wedge^\ell I_i \cong (\wedge^\ell \lambda)(\wedge^\ell I_i) = \wedge^\ell J_i$.
We get a short exact sequence of R-modules

$$0 \to \frac{\wedge^\ell X}{\mathrm{Tor}_R(\wedge^\ell X) + \wedge^\ell I_1 + \wedge^\ell I_2} \to \frac{\wedge^\ell F}{\wedge^\ell J_1 + \wedge^\ell J_2} \to \frac{R}{\mathrm{Fitt}_0(Y)} \to 0.$$

Corollary:

Suppose R in the previous Lemmas is a Noetherian UFD.
For $i = 1, 2$ we have an exact sequence of torsion modules

$$0 \to \mathrm{Tor}_R(X) \to X/I_i \xrightarrow{\bar{\lambda}} F/J_i \to Y \to 0.$$

Let RL_i be the first Chern class ideal of the torsion module X/I_i.
Let $R\theta_0$ be the first Chern class ideal of $\mathrm{Tor}_R(X)$.
Let $R\theta_1$ be the first Chern class ideal of Y.

Then θ_0 divides L_i in R, and $R(\theta_1 L_i/\theta_0)$ is the first Chern class ideal of both F/J_i and $\wedge^\ell F/\wedge^\ell J_i$. We have isomorphisms

$$N \overset{def}{=} \frac{\wedge^\ell F}{\wedge^\ell J_1 + \wedge^\ell J_2} \cong \frac{R}{R(\theta_1 L_1/\theta_0) + R(\theta_1 L_2/\theta_0)} \cong \frac{\theta_0 R}{R\theta_1 L_1 + R\theta_1 L_2}.$$

Let $\theta = \mathrm{g.c.d.}(L_1, L_2)$. Then $\rho = \theta_1\theta/\theta_0 \in R$.
The maximal pseudo-null submodule of N is

$$\mathrm{Ps}(N) = \rho N \cong \frac{\theta_1 \theta R}{R\theta_1 L_1 + R\theta_1 L_2} \cong \frac{R}{R(L_1/\theta) + R(L_2/\theta)}.$$

We have an exact sequence of pseudo-null modules

$$0 \to \mathrm{Ps}\left(\frac{\wedge^\ell X}{\mathrm{Tor}_R(\wedge^\ell X) + \wedge^\ell I_1 + \wedge^\ell I_2}\right) \to \rho N \to \rho \cdot \frac{R}{\mathrm{Fitt}_0(Y)} \to 0.$$

Note: In the Iwasawa theoretic set up, Y is pseudo-null

with trivial first Chern class ideal, i.e. θ_1 is a unit.

Hence $\rho = \theta/\theta_0$ where $\theta = \mathrm{g.c.d.}(L_1, L_2)$. Moreover,

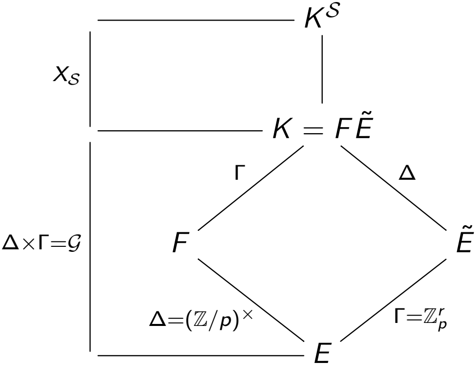
$$\rho N \cong \frac{R}{R(L_1/\theta) + R(L_2/\theta)} \cong \mathrm{Ps}\left(\frac{R}{RL_1 + RL_2}\right) = M'.$$

This means

$$c_2(M') = c_2\left(\mathrm{Ps}\left(\frac{\wedge^\ell X}{\mathrm{Tor}_R(\wedge^\ell X) + \wedge^\ell I_1 + \wedge^\ell I_2}\right)\right) + c_2\left(\theta/\theta_0 \cdot \frac{R}{\mathrm{Fitt}_0(Y)}\right)$$

Iwasawa theoretic set up.

Let E be a CM field in which a fixed prime $p > 2$ splits completely. Let K be the compositum of a finite abelian extension F of E of degree prime to p with the compositum $\tilde{E}$ of all the $\mathbb{Z}_p$-extensions of K. Let $\Omega = \mathbb{Z}_p[[\mathcal{G}]]$ when $\mathcal{G} = \mathrm{Gal}(K/E)$. For $\mathcal{S}$ a set of primes over p in E, let $X_{\mathcal{S}} = \mathrm{Gal}(K^{\mathcal{S}}/K)$ when $K^{\mathcal{S}}$ is the maximal abelian pro-p extension of K unramified outside of $\mathcal{S}$.



max. ab. pro p-ext. of K
unramified outside of $\mathcal{S}$

$\tilde{E}$ = compos. of all $\mathbb{Z}_p$-ext.s of E
$r = d + 1$ when Leopoldt conj. true

E = CM field, $[E : \mathbb{Q}] = 2d$

First Chern classes over CM fields.

Let $\psi : \Delta \to \overline{\mathbb{Q}}_p^*$ be a character.
Let $W = W(\overline{\mathbb{F}_p})$ be the ring of infinite Witt vectors over $\overline{\mathbb{F}}_p$.
Consider the ψ-isotypic components Ω^ψ and $X_\mathcal{S}^\psi$ and define

$$\begin{aligned} \Omega_W^\psi &= W\hat{\otimes}\Omega^\psi \cong W[[\Gamma]] \cong W[[t_1, \ldots, t_r]], \text{ and} \\ X_{\mathcal{S},W}^\psi &= W\hat{\otimes}X_\mathcal{S}^\psi. \end{aligned}$$

When $\mathcal{S}_1$ is a CM type over p, then one has a Katz p-adic L-function $\mathcal{L}_{\mathcal{S}_1,\psi}$ in the Iwasawa algebra Ω_W^ψ.

First Chern Class Main Conjecture:

$$c_1(X_{\mathcal{S}_1,W}^\psi) = c_1(\Omega_W^\psi/(\mathcal{L}_{\mathcal{S}_1,\psi})).$$

This was proved by Rubin when $[E : \mathbb{Q}] = 2$ and under some additional technical hypotheses for all CM fields E by work of Hida and Tilouine and of Hsieh.

Second Chern classes over CM fields.

$$\begin{tikzcd}[column sep=small]
 & & K^{\mathcal{S}} \arrow[dl, dash, "I_{\mathcal{T}_1}"'] \arrow[dr, dash, "I_{\mathcal{T}_2}"] \arrow[dd, dash, bend right=80, "X_{\mathcal{S}}"'] & & \\
 & K^{\mathcal{S}_1} \arrow[dr, dash, "X_{\mathcal{S}_1}"'] & & K^{\mathcal{S}_2} \arrow[dl, dash, "X_{\mathcal{S}_2}"] & \\
 & & K = F\tilde{E} \arrow[dl, dash, "\Gamma"'] \arrow[dr, dash, "\Delta"] \arrow[dd, dash, bend right=80, "\Delta\times\Gamma=\mathcal{G}"'] & & \\
 & F \arrow[dr, dash, "\Delta=(\mathbb{Z}/p)^{\times}"'] & & \tilde{E} \arrow[dl, dash, "\Gamma=\mathbb{Z}_p^r"] & \\
 & & E & &
\end{tikzcd}$$

max. ab. pro p-ext. of K
unramified outside of $\mathcal{S}$,
$\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2$

max. ab. pro p-ext.s of K
unram. outside of $\mathcal{S}_1$, resp. $\mathcal{S}_2$,
$\mathcal{S}_1, \mathcal{S}_2$ distinct CM types over p

$\tilde{E}$ = compos. of all $\mathbb{Z}_p$-ext.s of E
$r = d+1$ when Leopoldt conj. true

E = CM field, $[E : \mathbb{Q}] = 2d$

Theorem (BCGKST):

Let $\Omega_W^\psi \theta_0$ *be the first Chern class ideal of* $\mathrm{Tor}(X^\psi_{\mathcal{S},W})$.
Under the hypotheses of Hsieh's first Chern Class Main Conjecture, θ_0 *divides the g.c.d.* θ *of the L-functions* $\mathcal{L}_{\mathcal{S}_1,\psi}$ *and* $\mathcal{L}_{\mathcal{S}_2,\psi}$.

We have an equality of second Chern classes of pseudo-null modules

$$c_2\left(\frac{\Omega_W^\psi}{(\mathcal{L}_{\mathcal{S}_1,\psi}/\theta, \mathcal{L}_{\mathcal{S}_2,\psi}/\theta)}\right) = c_2\left(\mathrm{Ps}\left(\frac{\wedge^\ell_{\Omega_W^\psi} X^\psi_{\mathcal{S},W}}{\mathrm{Tor}_{\Omega_W^\psi}(\wedge^\ell_{\Omega_W^\psi} X^\psi_{\mathcal{S},W}) + \wedge^\ell_{\Omega_W^\psi} I^\psi_{\mathcal{T}_1,W} + \wedge^\ell_{\Omega_W^\psi} I^\psi_{\mathcal{T}_2,W}}\right)\right)$$

$$+ c_2\left(\theta/\theta_0 \cdot \frac{\Omega_W^\psi}{\mathrm{Fitt}_0\left(\mathrm{Ext}^2_{\Omega_W^\psi}(X^{\omega\psi^{-1}}_{\mathcal{S}^c,W}, \Omega_W^\psi)(1)\right)}\right)$$

where $\mathrm{Ext}^2_{\Omega_W^\psi}(X^{\omega\psi^{-1}}_{\mathcal{S}^c,W}, \Omega_W^\psi)(1)$ *is pseudo-null.*