Noncommutative Factorial Algebras

Milen Yakimov (LSU)
Maurice Auslander International Conference 2016

April 28, 2016

Main Results on Commutative UFDs

An **integral domain** is a **Unique Factorization Domain** (**UFD**, **Factorial Ring**) if every nonzero element is a product of primes in a unique way.

- Ex: $\mathbb{Z}$. More generally, every **Principle Ideal Domain** is a UFD.

Theorem [Gauss]

R is a UFD then $R[x]$ is a UFD.

Theorem [Auslander–Buchsbaum] 1959

Every regular local ring is a UFD.

Factorial varieties in Lie Theory

Coordinate rings in Lie Theory that are factorial:

[Popov] The coordinate rings of semisimple algebraic groups in char 0.
[Hochster] The homogeneous coordinate rings of Grassmannians.
[Kac-Peterson] The coordinate rings of Kac–Moody groups.

Reformulation for Noetherian Rings

Lemma [Nagata] 1958.

A noetherian integral domain R is a UFD if and only if every nonzero prime ideal contains a prime element.

Proof. $\Leftarrow$ Let $x \in R$ be a nonzero, nonunit and P be a minimal prime over (x). By Krull's principal ideal theorem, P has height 1. However it needs to contain a prime element $p \in P$, thus,

$$P = (p) \quad \text{and, so,} \quad x \in (p).$$

Therefore, $x = px'$ and we can continue by induction, using noetherianity.

Definitions

Let R be a noetherian domain, generally noncommutative.

Definition [Chatters 1983]

- A nonzero, nonunit element $p \in R$ is **prime** if $pR = Rp$ and R/pR is a domain.
- R is called a **noetherian UFD** if every nonzero prime ideal of R contains a homogeneous prime element.

Two prime elements $p, p' \in R$ are **associates** if $p' = up$ for a unit u.

Unique Factorization

An element $a \in R$ is called **normal** if $Ra = aR$. E.g., all **central** elements are normal.

Proposition

Every nonzero normal element of a noncommutative UFD has a unique factorization into primes up to reordering and associates.

Proof. The same as in the commutative case using the noncommutative principal ideal theorem: For every nonzero, nonunit normal element $a \in R$, a minimal prime over Ra has height 1.

The semi-center of universal enveloping algebras

Definition

The semi-center of $U(\mathfrak{g})$ is the direct sum $C(U(\mathfrak{g})) = \oplus_{\lambda \in \mathfrak{g}^*} C_\lambda(U(\mathfrak{g}))$, where for a character λ of $\mathfrak{g}$,

$$C_\lambda(U(\mathfrak{g})) := \{a \in U(\mathfrak{g}) \mid [x, a] = \lambda(x)a, \quad \forall x \in \mathfrak{g}\}.$$

The center of $U(\mathfrak{g})$ is $Z(U(\mathfrak{g})) = C_0(U(g))$. If $\mathfrak{g}$ is semisimple or nilpotent, then the semi-center of $U(\mathfrak{g})$ coincides with its center.
Example. Consider the Borel subalgebra $\mathfrak{b}$ of $\mathfrak{sl}_2$. It is spanned by H and E and $[H, E] = 2E$. Its semi-senter is $\mathbb{K}[E]$:

$$[H, \sum_n p_n(H)E^n] = \sum_n 2np_n(H)E^n, \quad [E, H^k E^n] = -\sum_{i=0}^{k-1} H^{k-1-i}(H-2)^i E^{n+1}.$$

Factoriality in the solvable case

Proposition

The normal elements of $U(\mathfrak{g})$ are $\cup_{\lambda \in \mathfrak{g}^*} C_\lambda(U(\mathfrak{g}))$.

Example. The normal elements of the 2-dim Borel subalgebra $\mathfrak{b}$ are $\{\mathbb{K}E^n \mid n \in \mathbb{N}\}$. There is only one prime element E.

Warning: Our goal is not to produce a theory with too few primes!

Theorem [Chatters]

For every solvable Lie algebra $\mathfrak{g}$ over an algebraically closed field of characteristic 0, $U(\mathfrak{g})$ is a UFD.

Proof of factoriality

Proof. Let J be any nonzero (two-sided) ideal of $U(\mathfrak{g})$. The adjoint action of $\mathfrak{g}$ on $U(\mathfrak{g})$ is locally finite, so J is a locally finite representation of $\mathfrak{g}$. By Lie's theorem there exits a $\mathfrak{g}$-eigenvector,

$$a \in J \cap C_\lambda(U(\mathfrak{g})), \quad a \neq 0.$$

Since $\mathfrak{g}$ is solvable, all prime ideals of $U(\mathfrak{g})$ are completely prime [Dixmier]. If J is prime, then it should contain an irreducible element a of the semi-center. One completes the proof by showing that $U(\mathfrak{g})/aU(\mathfrak{g})$ is a domain, so a is a prime element.

Comparison with Gauss' Lemma

• For an algebra B, $\sigma \in \mathrm{Aut}(B)$ and a skew-derivation δ, denote the skew-polynomial extension $B[x; \sigma, \delta]$.
• For a solvable Lie algebra $\mathfrak{b}$, there exists a chain of ideals

$$\mathfrak{b} = \mathfrak{b}_n \rhd \mathfrak{b}_{n-1} \rhd \ldots \rhd \mathfrak{b}_1 \rhd \mathfrak{b}_0 = \{0\} \text{ with } \dim(\mathfrak{b}_i/\mathfrak{b}_{i-1}) = 1.$$

Choosing $x_k \in \mathfrak{b}_k$, $x_k \notin \mathfrak{b}_{k-1}$, gives

$$\mathcal{U}(\mathfrak{b}) \cong \mathbb{K}[x_1][x_2; \mathrm{id}, \delta_2] \ldots [x_n; \mathrm{id}, \delta_n]$$

where all derivations $\delta_k = \mathrm{ad}_{x_k}$ are locally finite (locally nilpotent, if $\mathfrak{b}$ is nilpotent).

The factoriality of $U(\mathfrak{b})$ is a generalization of the Gauss Lemma.
Warning: It is easy to construct skew-polynomial extensions that are not factorial!

Quantum Groups

• $\mathfrak{g}$ a simple Lie algebra (more generally, a symmetrizable Kac–Moody algebra), G the corresponding simply connected group.
• $U_q(\mathfrak{g})$ the quantized univ env algebra, Chevalley generators $E_i, F_i, K_i^{\pm 1}$; $R_q[G]$ quantum function algebra.
• Lusztig's braid group action on $U_q(\mathfrak{g})$; T_w, $w \in W$ (Weyl group).
• **quantum Schubert cell algebras**, **quantum unipotent groups**

$$U_q(\mathfrak{n}_+ \cap w(\mathfrak{n}_-)) := U_q(\mathfrak{n}_+) \cap T_w(U_q(\mathfrak{n}_-)), \quad w \in W.$$

defined by Lusztig, De Concini–Kac–Procesi.
• **quantum double Bruhat cells**

$$R_q[G^{w,u}], \quad G^{w,u} := B_+ w B_+ \cap B_- u B_-, \quad w, u \in W.$$

Spectra of Quantum Groups

Early 90's, Hodges–Levasseur and Joseph did fundamental work on $\mathrm{Spec} R_q[G]$, aim: extend Dixmier's orbit method to quantum groups.

Conjecture. $\exists$ a homeomorphism $\mathrm{Dix}_G : \mathrm{Symp}(G, \pi) \xrightarrow{\cong} \mathrm{Prim} R_q[G]$.

Theorem [Joseph, Hodges–Levasseur–Toro, 1992]

For each simple group G:

- The H-prime ideals of $R_q[G]$ are indexed by $W \times W$: $I_{w,u}$ explicit in terms of Demazure modules of $U_q(\mathfrak{g})$.
- $\mathrm{Spec} R_q[G] \cong \bigsqcup_{w,u \in W} \mathrm{Spec} R_q[G^{w,u}]$ and

$$\mathrm{Spec} R_q[G^{w,u}] \cong \mathrm{Spec} Z(\mathrm{Spec} R_q[G^{w,u}]) \cong \text{ a torus.}$$

Conjecture wide open, but a bijective H-equivariant Dix_G constructed [2012].

Definition of quantum nilpotent algebras

Definition [Cauchon–Goodearl–Letzter] CGL Extensions (late 90's)

A **quantum nilpotent algebra** is a $\mathbb{K}$-algebra with an action of a torus H having the form

$$R := \mathbb{K}[x_1][x_2; (h_2\cdot), \delta_2] \cdots [x_N; (h_N\cdot), \delta_N]$$

for some $h_k \in H$, satisfying the following conditions:

- all δ_k are locally nilpotent $(h_k\cdot)$-derivations,
- the elements x_k are H-eigenvectors and the eigenvalues $h_k \cdot x_k = \lambda_k x_k$ are not roots of unity.

[G–L]: $H\text{–}\mathrm{Spec} R$ finite and a decomposition of $\mathrm{Spec} R$ into tori.
[C]: structure of H-primes of R.

Lie theory examples

- **Quantum Schubert cell algebras** $U_q(\mathfrak{n}_+ \cap w(\mathfrak{n}_-))$: a reduced expression $w = s_{i_1} \dots s_{i_N}$, the roots of $\mathfrak{n}_+ \cap w(\mathfrak{n}_-)$ are
$$\beta_1 = \alpha_{i_1},\ \beta_2 = s_{i_1}(\alpha_{i_2}),\ \dots,\ \beta_k = s_{i_1} \dots s_{i_{N-1}}(\alpha_{i_N}).$$
Presentation of $U_q(\mathfrak{n}_+ \cap w(\mathfrak{n}_-))$ by adjoining Lusztig's root vectors
$$E_{\beta_1} = E_{i_1}, E_{\beta_2} = T_{s_{i_1}}(E_{i_2}), \dots, E_{\beta_N} = T_{s_{i_1} \dots s_{i_{N-1}}}(E_{i_N}).$$
- **Quantum Weyl algebras**.
- **Quantum double Bruhat cells** (nontrivial presentation)
$$R_q[G^{w,u}] = (\mathcal{U}_q(\mathfrak{n}_- \cap w(n_+))^{\mathrm{op}} \bowtie \mathcal{U}_q(\mathfrak{n}_+ \cap u(\mathfrak{n}_-))[E^{-1}].$$

UFD property

Theorem [Launois–Lenagan–Rigal] (2005-2006)

- All quantum nillpotent algebras are UFDs.
- $R_q[G]$ is a UFD for all every complex simple group G.

Technical point: The precise statement in the first part is that every quantum nilpotent algebra R is an H-UFD (every nonzero H-prime ideal of R contains a homogeneous prime element). Furthermore, R is a UFD provided that it is torsionfree (the subgroup of $\mathbb{K}^*$ generated by the eigenvalues $\{\lambda_{kj} \mid k > j\}$ is torsionfree, where $h_k \cdot x_j = \lambda_{kj} x_j$).

Definitions on Cluster Algebras

[Fomin–Zelevinsky, 2001] A **cluster algebra** R is

- generated by an infinite set of generators grouped into embedded polynomial algebras $\mathbb{K}[y_1, \ldots, y_N] \subset R \subseteq \mathbb{K}[y_1^{\pm 1}, \ldots, y_N^{\pm 1}]$, **clusters**.
- Its clusters are obtained from each other by successive mutations $(y_1, \ldots, y_{k-1}, y_k, y_{k+1}, \ldots, y_N) \mapsto (y_1, \ldots, y_{k-1}, y'_k, y_{k+1}, \ldots, y_N)$

$$y'_k = \frac{\mathrm{monomial}_1 + \mathrm{monomial}_2}{y_k}$$

where $\gcd(\mathrm{monomial}_1, \mathrm{monomial}_2) = \gcd(y_k, \mathrm{monomial}_i) = 1$.

A **quantum cluster algebra** R: replace polynomial rings by quantum tori $\mathbb{K}\langle y_1, \ldots, y_N \rangle / (y_j y_k - q_{jk} y_k y_j)$, $q_{jk} \in \mathbb{K}^\times$.

Exact powers irrelevant will be just **powers in unique factorizations**.

Clusters on Quantum Nilpotent Algebras

Theorem [Goodearl-Y] (2014)

R = an arbitrary quantum nilpotent algebra. Chain of subalgebras $R_1 \subset R_2 \subset \ldots \subset R_N$.

- Each R_k has a unique homogeneous (under H) prime element y_k that does not belong to R_{k-1}.
- Under mild conditions, each such quantum nilpotent algebra R has a quantum cluster algebra structure with initial cluster $(y_1, \ldots, y_N)$.
- For $\tau \in S_N$, adjoin the generators of R in the order $x_{\tau(1)}, \ldots, x_{\tau(N)}$. Chain of subalgebras $R_{\tau,1} \subset R_{\tau,2} \subset \ldots \subset R_{\tau,N}$. The sequence of primes $(y_{\tau,1}, \ldots, y_{\tau,N})$ is another cluster Σ_τ.
- The cluster algebra R is generated by the primes in the finitely many clusters Σ_τ for $\tau \in S_N$.

Commutative UFDs in Cluster Algebra setting in Geiss-Leclerc–Schröer but too many primes, no idea which ones are cluster variables.

Applications

Berenstein–Zelevinsky Conjecture [Goodearl-Y 2016]

For all complex simple Lie groups G and Weyl groups elements w and u, the quantized coordinate ring of the double Bruhat cell $R_q[G^{w,u}]$ has a canonical cluster algebra structure.

Theorem [GY, 2014]

For all symmetrizable Kac–Moody algebras $\mathfrak{g}$ and Weyl group elements w, $U_q(\mathfrak{n}_+ \cap w(\mathfrak{n}_-))$ has a cluster algebra structure.

Previously proved by Geiss–Leclerc–Schröer for symmetric Kac–Moody algebras $\mathfrak{g}$.
Other Applications: Quantum Weyl algebras.

Maximal green sequences I

Notation for the elements of S_N: $\tau = [\tau(1), \dots, \tau(N)]$.

Procedure. Pull the number 1 all the way to the right (preserving the order of the other numbers), then pull the number 2 to the right just after the N, ..., at the end pull the number $N-1$ to the right after the N:

$$\begin{aligned} \mathrm{id} = &[\textcircled{1}, \textcircled{2}, 3, \dots, N] \mapsto \dots \mapsto \\ &[\textcircled{2}, 3, \dots, N, \textcircled{1}] \mapsto \dots \mapsto \\ &[3, \dots, N, \textcircled{2}, \textcircled{1}] \mapsto \cdots \mapsto \\ &[N-1, N, \dots, 2, 1] \mapsto \dots \mapsto \\ &[N, N-1, \dots, 2, 1] = w_\circ. \end{aligned}$$

Maximal green sequences II

Theorem [Y]

For each quantum nilpotent algebra R the sequence of clusters $\Sigma_{\mathrm{id}} \to \ldots \to \Sigma_{w_\circ}$ is a maximal green sequence of mutations of length

$$\binom{n_1}{2} + \cdots + \binom{n_1}{2}$$

At each step the two clusters are either related by a one-step mutation or are identical.

Applying, results of Keller, gives a formula for the Donaldson–Thomas invariant of the corresponding 3-Calabi–Yau category.

Notes: (1) These cluster algebras are of **very** infinite type!
(2) No explicit mutation of quivers in the proof. **Red/green vertices** come from **positive/negative powers** of factorizations into primes.

Categorifications

- **Abelian Categorifications of Cluster Algebras**: Initiated by Buan, Marsh, Reineke, Reiten and Todorov.
- **Monoidal Categorifications of Cluster Algebras**: Axiomatized by Hernandez–Leclerc.
- **Explicit Abelian Categorifications** of $U_q(\mathfrak{n}_+ \cap w(\mathfrak{n}_-))$ constructed by Geiss–Leclerc–Schröer, $\mathfrak{g}=$ symmetric KM.
- **Monoidal Categorifications** of $U_q(\mathfrak{n}_+ \cap w(\mathfrak{n}_-))$ by Kang–Kashiwara–Kim–Oh (representations of Khovanov–Lauda–Rouquier algebras) and Qin, $\mathfrak{g}=$ symmetric KM.
- **Problem**. Construct explicit Abelian and Monoidal Categorifications for all quantum nilpotent algebras. Various applications to Lie theory (canonical bases).
  Role of factoriality of the algebra? Trick: establish properties of particular sequence of mutations (e.g. a green sequence), and then prove that this implies properties of all mutations.

Poisson UFDs

A similar concept of **Poisson UFDs**. In the case of coordinate rings, geometric methods using **Poisson manifolds**.

Applications to **Discriminants** of orders in central simple algebras and **Cluster Algebras**.

Many classes of examples, based on **Poisson Lie groups** and **Poisson homogeneous spaces**.