Noncommutative Factorial Algebras

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An integral domain is a Unique Factorization Domain (UFD, Factorial Ring) if every nonzero element is a product of primes in a unique way.

- Ex: \( \mathbb{Z} \). More generally, every Principle Ideal Domain is a UFD.

**Theorem [Gauss]**

\( R \) is a UFD then \( R[x] \) is a UFD.

**Theorem [Auslander–Buchsbaum] 1959**

Every regular local ring is a UFD.
Factorial varieties in Lie Theory

Coordinate rings in Lie Theory that are factorial:

[Popov] The coordinate rings of semisimple algebraic groups in char 0.
[Hochster] The homogeneous coordinate rings of Grassmannians.
[Kac-Peterson] The coordinate rings of Kac–Moody groups.
A noetherian integral domain $R$ is a UFD if and only if every nonzero prime ideal contains a prime element.

Proof. $\iff$ Let $x \in R$ be a nonzero, nonunit and $P$ be a minimal prime over $(x)$. By Krull’s principal ideal theorem, $P$ has height 1. However it needs to contain a prime element $p \in P$, thus,

$$P = (p)$$

and, so, $x \in (p)$.

Therefore, $x = px'$ and we can continue by induction, using noetherianity.
Let $R$ be a noetherian domain, generally noncommutative.

**Definition [Chatters 1983]**

- A nonzero, nonunit element $p \in R$ is **prime** if $pR = Rp$ and $R/pR$ is a domain.
- $R$ is called a **noetherian UFD** if every nonzero prime ideal of $R$ contains a homogeneous prime element.

Two prime elements $p, p' \in R$ are **associates** if $p' = up$ for a unit $u$. 
An element \( a \in R \) is called **normal** if \( Ra = aR \). E.g., all **central** elements are normal.

**Proposition**

Every nonzero normal element of a noncommutative UFD has a unique factorization into primes up to reordering and associates.

**Proof.** The same as in the commutative case using the noncommutative principal ideal theorem: For every nonzero, nonunit normal element \( a \in R \), a minimal prime over \( Ra \) has height 1.
The semi-center of universal enveloping algebras

Definition

The semi-center of $U(g)$ is the direct sum $C(U(g)) = \bigoplus_{\lambda \in g^*} C_{\lambda}(U(g))$, where for a character $\lambda$ of $g$,

$$C_{\lambda}(U(g)) := \{ a \in U(g) \mid [x, a] = \lambda(x)a, \quad \forall x \in g \}.$$

The center of $U(g)$ is $Z(U(g)) = C_0(U(g))$. If $g$ is semisimple or nilpotent, then the semi-center of $U(g)$ coincides with its center.

Example. Consider the Borel subalgebra $\mathfrak{b}$ of $\mathfrak{sl}_2$. It is spanned by $H$ and $E$ and $[H, E] = 2E$. Its semi-center is $\mathbb{K}[E]$:

$$[H, \sum_n p_n(H)E^n] = \sum_n 2np_n(H)E^n, \quad [E, H^k E^n] = -\sum_{i=0}^{k-1} H^{k-1-i}(H-2)^i E^{n+1}.$$
Factoriality in the solvable case

Proposition
The normal elements of $U(g)$ are $\bigcup_{\lambda \in g^*} C_{\lambda}(U(g))$.

Example. The normal elements of the 2-dim Borel subalgebra $b$ are $\{KE^n \mid n \in \mathbb{N}\}$. There is only one prime element $E$.

Warning: Our goal is not to produce a theory with too few primes!

Theorem [Chatters]
For every solvable Lie algebra $g$ over an algebraically closed field of characteristic 0, $U(g)$ is a UFD.
Proof of factoriality

**Proof.** Let $J$ be any nonzero (two-sided) ideal of $U(\mathfrak{g})$. The adjoint action of $\mathfrak{g}$ on $U(\mathfrak{g})$ is locally finite, so $J$ is a locally finite representation of $\mathfrak{g}$. By Lie’s theorem there exists a $\mathfrak{g}$-eigenvector,

$$a \in J \cap C_\lambda(U(\mathfrak{g})), \quad a \neq 0.$$

Since $\mathfrak{g}$ is solvable, all prime ideals of $U(\mathfrak{g})$ are completely prime [Dixmier]. If $J$ is prime, then it should contain an irreducible element $a$ of the semi-center. One completes the proof by showing that $U(\mathfrak{g})/aU(\mathfrak{g})$ is a domain, so $a$ is a prime element.
Comparison with Gauss’ Lemma

• For an algebra $B$, $\sigma \in \text{Aut}(B)$ and a skew-derivation $\delta$, denote the skew-polynomial extension $B[x; \sigma, \delta]$.
• For a solvable Lie algebra $b$, there exists a chain of ideals

$$b = b_n \supset b_{n-1} \supset \ldots \supset b_1 \supset b_0 = \{0\} \text{ with } \dim(b_i/b_{i-1}) = 1.$$ 

Choosing $x_k \in b_k$, $x_k \notin b_{k-1}$, gives

$$U(b) \cong \mathbb{K}[x_1][x_2; \text{id}, \delta_2] \ldots [x_n; \text{id}, \delta_n]$$

where all derivations $\delta_k = \text{ad}_{x_k}$ are locally finite (locally nilpotent, if $b$ is nilpotent).

The factoriality of $U(b)$ is a generalization of the Gauss Lemma.

**Warning:** It is easy to construct skew-polynomial extensions that are not factorial!
Quantum Groups

- \( \mathfrak{g} \) a simple Lie algebra (more generally, a symmetrizable Kac–Moody algebra), \( G \) the corresponding simply connected group.
- \( U_q(\mathfrak{g}) \) the quantized univ env algebra, Chevalley generators \( E_i, F_i, K_i^{\pm1} \); \( R_q[G] \) quantum function algebra.
- Lusztig’s braid group action on \( U_q(\mathfrak{g}); T_w, w \in W \) (Weyl group).
- quantum Schubert cell algebras, quantum unipotent groups

\[
U_q(n_+ \cap w(n_-)) := U_q(n_+) \cap T_w(U_q(n_-)), \quad w \in W.
\]

defined by Lusztig, De Concini–Kac–Procesi.
- quantum double Bruhat cells

\[
R_q[G^{w,u}], \quad G^{w,u} := B_+ w B_+ \cap B_- u B_-, \quad w, u \in W.
\]
Early 90’s, Hodges–Levasseur and Joseph did fundamental work on \( \text{Spec} R_q[G] \), aim: extend Dixmier’s orbit method to quantum groups.

**Conjecture.** \( \exists \) a homeomorphism \( \text{Dix}_G : \text{Symp}(G, \pi) \xrightarrow{\cong} \text{Prim} R_q[G] \).

**Theorem [Joseph, Hodges–Levasseur–Toro, 1992]**

For each simple group \( G \):
- The \( H \)-prime ideals of \( R_q[G] \) are indexed by \( W \times W \): \( I_{w,u} \) explicit in terms of Demazure modules of \( U_q(g) \).
- \( \text{Spec} R_q[G] \cong \bigsqcup_{w,u \in W} \text{Spec} R_q[G^{w,u}] \) and
  \[
  \text{Spec} R_q[G^{w,u}] \cong \text{Spec} \mathbb{Z}(\text{Spec} R_q[G^{w,u}]) \cong \text{a torus}.
  \]

Conjecture wide open, but a bijective \( H \)-equivariant \( \text{Dix}_G \) constructed [2012].
Definition [Cauchon–Goodearl–Letzter] CGL Extensions (late 90’s)

A quantum nilpotent algebra is a $\mathbb{K}$-algebra with an action of a torus $H$ having the form

$$R := \mathbb{K}[x_1][x_2; (h_2 \cdot), \delta_2] \cdots [x_N; (h_N \cdot), \delta_N]$$

for some $h_k \in H$, satisfying the following conditions:

- all $\delta_k$ are locally nilpotent $(h_k \cdot)$-derivations,
- the elements $x_k$ are $H$-eigenvectors and the eigenvalues $h_k \cdot x_k = \lambda_k x_k$ are not roots of unity.

[G–L]: $H$–$\text{Spec} R$ finite and a decomposition of $\text{Spec} R$ into tori.
[C]: structure of $H$-primes of $R$. 

Definition of Quantum Nilpotent Algebras
Lie theory examples

- **Quantum Schubert cell algebras** $U_q(n_+ \cap w(n_-))$: a reduced expression $w = s_{i_1} \ldots s_{i_N}$, the roots of $n_+ \cap w(n_-)$ are
  \[ \beta_1 = \alpha_{i_1}, \beta_2 = s_{i_1}(\alpha_{i_2}), \ldots, \beta_k = s_{i_1} \ldots s_{i_{N-1}}(\alpha_{i_N}) . \]

  Presentation of $U_q(n_+ \cap w(n_-))$ by adjoining Lusztig’s root vectors
  \[ E_{\beta_1} = E_{i_1}, E_{\beta_2} = T_{s_{i_1}}(E_{i_2}), \ldots, E_{\beta_N} = T_{s_{i_1} \ldots s_{i_{N-1}}}(E_{i_N}) . \]

- **Quantum Weyl algebras.**

- **Quantum double Bruhat cells** (nontrivial presentation)
  \[ R_q[G^{w,u}] = (U_q(n_- \cap w(n_+))^\text{op} \rtimes U_q(n_+ \cap u(n_-)))[E^{-1}] . \]

- All quantum nilpotent algebras are UFDs.
- $R_q[G]$ is a UFD for all every complex simple group $G$.

**Technical point**: The precise statement in the first part is that every quantum nilpotent algebra $R$ is an $H$-UFD (every nonzero $H$-prime ideal of $R$ contains a homogeneous prime element). Furthermore, $R$ is a UFD provided that it is torsionfree (the subgroup of $\mathbb{K}^*$ generated by the eigenvalues $\{\lambda_{kj} \mid k > j\}$ is torsionfree, where $h_k \cdot x_j = \lambda_{kj} x_j$).
[Fomin–Zelevinsky, 2001] A cluster algebra \( R \) is

- generated by an infinite set of generators grouped into embedded polynomial algebras \( \mathbb{K}[y_1, \ldots, y_N] \subset R \subset \mathbb{K}[y_1^{\pm 1}, \ldots, y_N^{\pm 1}] \), clusters.
- Its clusters are obtained from each other by successive mutations
  \( (y_1, \ldots, y_{k-1}, y_k, y_{k+1}, \ldots, y_N) \mapsto (y_1, \ldots, y_{k-1}, y'_k, y_{k+1}, \ldots, y_N) \)
  \[ y'_k = \frac{\text{monomial}_1 + \text{monomial}_2}{y_k} \]
  where \( \gcd(\text{monomial}_1, \text{monomial}_2) = \gcd(y_k, \text{monomial}_i) = 1 \).

A quantum cluster algebra \( R \): replace polynomial rings by quantum tori \( \mathbb{K}\langle y_1, \ldots, y_N \rangle/(y_j y_k - q_{jk} y_k y_j), q_{jk} \in \mathbb{K}^\times \).

Exact powers irrelevant will be just powers in unique factorizations.
Clustering on Quantum Nilpotent Algebras


Let $R$ be an arbitrary quantum nilpotent algebra. Chain of subalgebras $R_1 \subset R_2 \subset \ldots \subset R_N$.

- Each $R_k$ has a unique homogeneous (under $H$) prime element $y_k$ that does not belong to $R_{k-1}$.
- Under mild conditions, each such quantum nilpotent algebra $R$ has a quantum cluster algebra structure with initial cluster $(y_1, \ldots, y_N)$.
- For $\tau \in S_N$, adjoin the generators of $R$ in the order $x_{\tau(1)}, \ldots, x_{\tau(N)}$. Chain of subalgebras $R_{\tau,1} \subset R_{\tau,2} \subset \ldots \subset R_{\tau,N}$. The sequence of primes $(y_{\tau,1}, \ldots, y_{\tau,N})$ is another cluster $\Sigma_{\tau}$.
- The cluster algebra $R$ is generated by the primes in the finitely many clusters $\Sigma_{\tau}$ for $\tau \in S_N$.

Commutative UFDs in Cluster Algebra setting in Geiss-Leclerc–Schröer but too many primes, no idea which ones are cluster variables.
Applications

Berenstein–Zelevinsky Conjecture [Goodearl-Y 2016]

For all complex simple Lie groups $G$ and Weyl groups elements $w$ and $u$, the quantized coordinate ring of the double Bruhat cell $R_q[G^{w,u}]$ has a canonical cluster algebra structure.

Theorem [GY, 2014]

For all symmetrizable Kac–Moody algebras $\mathfrak{g}$ and Weyl group elements $w$, $U_q(n_+ \cap w(n_-))$ has a cluster algebra structure.

Previously proved by Geiss–Leclerc–Schröer for symmetric Kac–Moody algebras $\mathfrak{g}$.

Other Applications: Quantum Weyl algebras.
Maximal green sequences I

Notation for the elements of $S_N$: $\tau = [\tau(1), \ldots, \tau(N)]$.

**Procedure.** Pull the number 1 all the way to the right (preserving the order of the other numbers), then pull the number 2 to the right just after the $N$, ..., at the end pull the number $N - 1$ to the right after the $N$:

$$\text{id} = [1, 2, 3, \ldots, N] \mapsto \ldots \mapsto [2, 3, \ldots, N, 1] \mapsto \ldots \mapsto [3, \ldots, N, 2, 1] \mapsto \ldots \mapsto [N - 1, N, \ldots, 2, 1] \mapsto \ldots \mapsto [N, N - 1, \ldots, 2, 1] = w_\circ.$$
Maximal green sequences II

**Theorem [Y]**

For each quantum nilpotent algebra $R$ the sequence of clusters $\Sigma_{id} \rightarrow \ldots \rightarrow \Sigma_{w_0}$ is a maximal green sequence of mutations of length

$$\binom{n_1}{2} + \cdots + \binom{n_1}{2}$$

At each step the two clusters are either related by a one-step mutation or are identical.

Applying, results of Keller, gives a formula for the Donaldson–Thomas invariant of the corresponding 3-Calabi–Yau category.

**Notes:** (1) These cluster algebras are of **very** infinite type!
(2) No explicit mutation of quivers in the proof. **Red/green vertices** come from **positive/negative powers** of factorizations into primes.
Categorifications

- **Abelian Categorifications of Cluster Algebras**: Initiated by Buan, Marsh, Reineke, Reiten and Todorov.

- **Monoidal Categorifications of Cluster Algebras**: Axiomatized by Hernandez–Leclerc.

- **Explicit Abelian Categorifications** of $U_q(n_+ \cap \mathfrak{w}(n_-))$ constructed by Geiss–Leclerc–Schröer, $\mathfrak{g} = \text{symmetric KM}$.  

- **Monoidal Categorifications** of $U_q(n_+ \cap \mathfrak{w}(n_-))$ by Kang–Kashiwara–Kim–Oh (representations of Khovanov–Lauda–Rouquier algebras) and Qin, $\mathfrak{g} = \text{symmetric KM}$. 

- **Problem**. Construct explicit Abelian and Monoidal Categorifications for all quantum nilpotent algebras. Various applications to Lie theory (canonical bases). Role of factoriality of the algebra? Trick: establish properties of particular sequence of mutations (e.g. a green sequence), and then prove that this implies properties of all mutations.
A similar concept of **Poisson UFDs**. In the case of coordinate rings, geometric methods using **Poisson manifolds**.

Applications to **Discriminants** of orders in central simple algebras and **Cluster Algebras**.

Many classes of examples, based on **Poisson Lie groups** and **Poisson homogeneous spaces**.