Some representation theory arising from set-theoretic homological algebra

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There are only few categories of (possibly infinitely generated) modules all of whose modules decompose into direct sums of small submodules. The picture changes completely when direct sums are replaced by transfinite extensions. Starting from [2], many classes $\mathcal{C}$ of modules were shown to be deconstructible, that is, each of their modules expressible as a transfinite extension of small modules from $\mathcal{C}$. The deconstructibility implies existence of $\mathcal{C}$-precovers, hence makes $\mathcal{C}$ fit in the machinery of relative homological algebra [5], [6]. While deconstructible classes appeared to be ubiquitous, some important non-precovering classes of modules have gradually emerged, first under extra set-theoretic assumptions [4], and then in ZFC.

The class of all flat Mittag-Leffler modules over any non-perfect ring is an example of a non-precovering class [1]. Moreover, it is just the zero dimensional instance (for $T = R$ and $n = 0$) of non-precovering of the class of all locally T-free modules, where T is any n-tilting module which is not $\sum$-pure split. The phenomenon occurs even for finite dimensional algebras, e.g., when R is hereditary, of infinite representation type, and T is the Lukas tilting module.

A key tool in [1] is the construction of an appropriate tree module, [8]. Šaroch has recently generalized this construction in order to solve an old problem by Auslander on the existence of almost split sequences, cf. [2] and [7].

References

[1] L.Angeleri Hügel, J.Šaroch, J.Trlifaj, *Approximations and Mittag-Leffler conditions*, available at https://www.researchgate.net/publication/280494406 Approximations and Mittag-Leffler conditions.

[2] M.Auslander, *Existence theorems for almost split sequences*, Ring theory II (Proc. Conf. Univ. Oklahoma, 1975), 144. Lecture Notes in Pure and Appl. Math., Vol. 26, Dekker, New York, 1977.

[3] L. Bican, R. El Bashir, E. E. Enochs, *Every module has a flat cover*, Bull. London Math. Soc. 33(2001), 385-390.

[4] P. C. Eklof, S. Shelah, *On the existence of precovers*, Illinois J. Math. 47(2003), 173-188.

[5] E. E. Enochs, O. M. G. Jenda, *Relative Homological Algebra*, 2nd rev. ext. ed., Vols. 1 and 2, GEM 30 and 54, W. de Gruyter, Berlin 2011.

[6] R. Göbel, J. Trlifaj, *Approximations and Endomorphism Algebras of Modules*, 2nd rev. ext. ed., Vols. 1 and 2, GEM 41, W. de Gruyter, Berlin 2012.

[7] J. Šaroch, *On the non-existence of right almost split maps*, available at arXiv: 1504.01631v4.

[8] A. Slávik, J. Trlifaj, *Approximations and locally free modules*, Bull. London Math. Soc. 46(2014), 76-90.