Very Flat, Locally Very Flat, and Contraadjusted Modules

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27th April 2016

Introducing the classes

Throughout the whole talk R = commutative associative ring (with a unit), module = R-module.

$R[s^{-1}]$ = localization of R in the multiplicative set $\{1, s, s^2, \dots\}$

Definition (L. Positselski: Contraherent cosheaves, [arXiv:1209.2995])

A module C is called *contraadjusted* if for every $s \in R$,

$$\mathrm{Ext}^1_R(R[s^{-1}], C) = 0.$$

A module V is *very flat* if

$$\mathrm{Ext}^1_R(V, C) = 0$$

for every contraadjusted module C.

The origin of the classes

A bit of geometric motivation:

Theorem

If U, V are open affine subschemes of a scheme X satisfying $U \subseteq V$, then the $\mathcal{O}_X(V)$-module $\mathcal{O}_X(U)$ is very flat.

Cotorsion pair $(\mathcal{VF}, \mathcal{CA})$

We denote $\mathcal{VF}$ = class of all very flat modules, $\mathcal{CA}$ = all contraadjusted modules.

Directly from the definition, the classes in question form a cotorsion pair $(\mathcal{VF}, \mathcal{CA})$; since this pair is generated by a *set* (namely $\{R[s^{-1}] \mid s \in R\}$), by the well known machinery (recall the preceding talk!), there are automatically module approximations at our disposal: In particular, for each module M, there are $C \in \mathcal{CA}$ and $V \in \mathcal{VF}$, which fit into the exact sequence

$$0 \to M \to C \to V \to 0$$

(*special $\mathcal{CA}$-preenvelope* of M).
Similarly, for each module M we have the sequence

$$0 \to C \to V \to M \to 0$$

with $C \in \mathcal{CA}$, $V \in \mathcal{VF}$ (*special $\mathcal{VF}$-precover*).

Some examples

Some non-trivial examples in Abelian groups:

Example

As a group, $G = \mathbb{Z}[i][(2+i)^{-1}]$ is very flat (of rank 2); in fact, there is a non-split exact sequence

$$0 \to \mathbb{Z} \to G \to \mathbb{Z}[5^{-1}] \to 0.$$

Example

The torsion group

$$\bigoplus_{p \text{ prime}} \mathbb{Z}/p\mathbb{Z}$$

is contraadjusted, but not cotorsion.

Still searching for examples, i.e. from $\mathcal{VF} \cap \mathcal{CA}$.

Envelopes & Covers

The existence of envelopes and covers is neither rare, nor really common. Some examples:

- Injective envelopes (always exist)
- Cotorsion envelopes (always exist)
- Projective covers (only for perfect rings)
- Flat covers (always exist).

Recall:

Theorem (Enochs, Xu)

If the class $\mathcal{A}$ in the cotorsion pair $(\mathcal{A}, \mathcal{B})$ is closed under direct limits, then it is covering.

It is suspected (Enochs) that the converse is true as well.

Very flat covers

From now on, $R =$ Noetherian commutative ring.

Theorem (S.-Trlifaj)

Let R be a Noetherian ring. If the class $\mathcal{VF}$ is covering, then the spectrum of R is finite.

If further R is a domain, then the following are equivalent:

- *$\mathcal{VF}$ is a covering class.*
- *R has finite spectrum.*
- *Each flat module is very flat.*

The equivalence is most likely true for all Noetherian rings.

If R has finite spectrum, then its Krull dimension does not exceed 1.

Contraadjusted envelopes

Theorem (S.-Trlifaj)

Let R be a Noetherian ring. If the class $\mathcal{CA}$ is enveloping, then the spectrum of R is finite.

If further R is a domain, then the following are equivalent:

- *$\mathcal{CA}$ is an enveloping class.*
- *R has finite spectrum.*
- *Each contraadjusted module is cotorsion.*

Introducing locally very flat modules

Definition

We call a module M *locally very flat*, if M possesses a system $\mathcal{S}$ of countably presented very flat submodules such that

- $0 \in \mathcal{S}$,
- for each countable set $X \subseteq M$ there is $S \in \mathcal{S}$ satisfying $X \subseteq S$,
- $\mathcal{S}$ is closed under unions of countable chains.

$\mathcal{LV}$ = class of all locally very flat modules.

An analogous class is formed by the *flat Mittag-Leffler modules* (from the preceding talk!), which are obtained by the replacement "very flat" $\rightarrow$ "projective" in the definition above. $\mathcal{FM}$ = class of all flat Mittag-Leffler modules.

Similarities between $\mathcal{LV}$ and $\mathcal{FM}$

For Dedekind domains, we know a bit more about the class $\mathcal{LV}$ (an analog of so-called Pontryagin criterion):

Theorem (S.-Trlifaj)

Let R be a Dedekind domain. The following are equivalent for a module M:

- $M \in \mathcal{LV}$,
- *For every finite set $F \subseteq M$, there is a countable generated very flat pure submodule $V \subseteq M$ with $F \subseteq V$.*
- *Each finite rank submodule of M is very flat.*

Approximation properties of $\mathcal{LV}$

Flat Mittag-Leffler modules form a well-known "pathological" class: Although it "looks like" a left class in a cotorsion pair, it is not precovering for non-perfect rings (Angeleri-Šaroch-Trlifaj 2014).

The analogy we have for locally very flat modules is the following:

Theorem (S.-Trlifaj)

For a Noetherian ring R, if the class $\mathcal{LV}$ is precovering, then the spectrum of R is finite.

For R a domain, the reverse implication holds (plus all the other equivalent conditions).

The End

More to be found at [arXiv:1601.00783].

Questions? Comments?