The Jacobson-Toeplitz Algebra and Direct Finiteness

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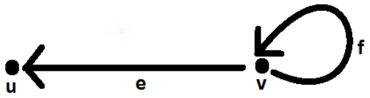
Notation and Definitions

We'll pick a few conventions and stick to them throughout:

1. $\mathbb{K}$ is a field of arbitrary characteristic.
2. All modules are left modules.
3. $R = \mathbb{K}\langle x, y\rangle/(xy - 1)$ is the **Jacobson-Toeplitz Algebra.** $I = \mathrm{Soc}(R)$.
4. Γ is the quiver:



Note that $R \cong L_{\mathbb{K}}(\Gamma)$, the Leavitt path algebra of Γ.

Previous Work

Theorem

Let I denote the socle of R. Then the following hold:

1. *I can be written as $I = \bigoplus_{i=1}^{\infty} S_i$, where each $S_i = R(y^{i-1}x^{i-1} - y^i x^i)$ is a faithful simple R-module.*
2. *$S_i \cong S_1$ for all $i \geq 1$. In fact, if we let $v_i = y^i(1 - yx)$ for all $i \geq 0$, then $\{v_i\}_{i \geq 0}$ is a $\mathbb{K}$-basis for S_1, with $yv_i = v_{i+1}$, $xv_{i+1} = v_i$, and $xv_0 = 0$ for all $i \geq 0$.*
3. *I is the two-sided ideal generated by $1 - yx$, and is the unique minimal two-sided ideal of R.*

Comments

1. See [Alahmedi et. al. 2013], [Bavula 2010], [Colak 2011].
2. $R/I \cong \mathbb{K}[x, x^{-1}]$.
3. As a module over $\mathbb{K}[x] \subset R$, S_1 is the injective hull of $\mathbb{K}[x]/(x)$.

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Previous Work (Cont.)

From the Leavitt Path Algebra Literature

1. Simple modules: S_1 and $\mathbb{K}[x, x^{-1}]/(p(x))$, where $p(x)$ is an irreducible element of $\mathbb{K}[x, x^{-1}]$ [Ara, Rang. 2014].
2. R is left hereditary [Ara et. al 2007].
3. The module of finitely-generated projectives is generated by R and S_1, with the relation $R \oplus S_1 \cong R$ [Ara et. al. 2007].
4. Ext groups between Chen modules are known [Abrams et. al. 2015].
5. The two-sided ideals of R can be computed [Colak 2011].

Left Ideals of R

Theorem (Iovanov, Sistko 2016)

Every left ideal of R can be written as $Rp(x) \oplus \Sigma$, where $p(x)$ is a monic polynomial and Σ is contained in the socle I. There are canonical choices for $p(x)$ and Σ.

Comments

1. $p(x)$ is unique if chosen of minimal degree (note that $p \equiv 0$ if and only if the left ideal is semisimple.)
2. Σ is determined by its socle as a $\mathbb{K}[x]$-module.
3. Since R is hereditary, this classifies arbitrary projectives.
4. Corollary: Every left ideal is either semisimple or finitely generated.

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Finite-Length Modules and Ext Spaces

Theorem (Iovanov, Sistko 2016)

Let M be a finite-length R-module. Then the following hold:

1. *M is the middle term of a short exact sequence $0 \to S_1^{\oplus k} \to M \to F \to 0$, for some $k \in \mathbb{N}$ and finite-dimensional R-module F.*
2. *Let p be a (not necessarily irreducible) Laurent polynomial in x. Then $\operatorname{Ext}^1(\mathbb{K}[x, x^{-1}]/(p), S_1) \cong \mathbb{K}[T]/(p^*(T))$, where p^* is the polynomial defined by $p^*(y) = p(x)y^{\deg(p)} \in \mathbb{K}[y] \subseteq R$.*

Comments

1. Extends results of [Abrams et. al. 2015].
2. Can use the fact that R is hereditary to get formulas for $\dim_{\mathbb{K}} \operatorname{Ext}^1(M, N)$.

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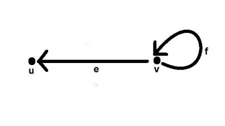
The Category WSP

1. Any R-module M fits into a short exact sequence $0 \to IM \xrightarrow{\sigma} M \xrightarrow{\pi} M/IM \to 0$. Note that IM is semisimple projective, hence injective as a $\mathbb{K}[x]$-module.
2. **Objects of WSP:** Pairs (M, α), $\alpha : M/IM \to M$ a $\mathbb{K}[x]$-module morphism with $\pi \circ \alpha = \mathrm{id}_{M/IM}$.
3. **Morphisms of WSP:** $(M, \alpha) \to (N, \beta)$ is an R-module morphism $\varphi : M \to N$ with $\mathrm{Im}(\varphi \circ \alpha) \subset \beta$.

The Category LRep(Γ)

The full subcategory of representations of Γ:



on which f acts as an invertible map.

An Equivalence of Categories (Cont.)

Theorem (Iovanov, Sistko 2016)

The categories WSP *and* $\mathrm{LRep}(\Gamma)$ *are equivalent.*

Comments

1. $\mathrm{LRep}(\Gamma)$ is just the category of representations of $\mathbb{K}\Gamma[t]/(tf-1, ft-1)$.
2. Realizes the category of R-modules as a quotient of $\mathrm{LRep}(\Gamma)$.
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Kaplansky's Direct Finiteness Conjecture

Direct Finiteness Conjecture

Let G be a (countable discrete) group. If $a, b \in \mathbb{K}G$ satisfy $ab = 1$, then $ba = 1$ as well.

Known Results

1. True if $\text{char}(\mathbb{K}) = 0$ [Montgomery 1969].
2. True in arbitrary characteristic for "finitely-generated residually finite"-by-sofic groups [Berlai 2015].
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Kaplansky's Direct Finiteness Conjecture (Cont.)

R and the DFC

1. Suppose that $a, b \in \mathbb{K}G$ satisfy $ab = 1$ but $ba \neq 1$. Then the map $R \to \mathbb{K}G$ taking $x \mapsto a$, $y \mapsto b$ is an injection.
2. $\mathbb{K}G$ then becomes a faithful representation of R.
3. Let Σ be the sum of all simple projective submodules of $\mathbb{K}G$, $F \supset \Sigma$ the R-submodule of $\mathbb{K}G$ such that F/Σ is the locally finite part of $\mathbb{K}G/\Sigma$.
4. $\Sigma \subset F \subset \mathbb{K}G$ is a filtration of left R-modules, and right $\mathbb{K}G$-modules.

Question

What sorts of G-representations must Σ, F, F/Σ, and $\mathbb{K}G/F$ be?

Gratitude Slide

Thanks for listening!

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