An Adjoint to the Auslander-Gruson-Jensen Functor

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Main Result

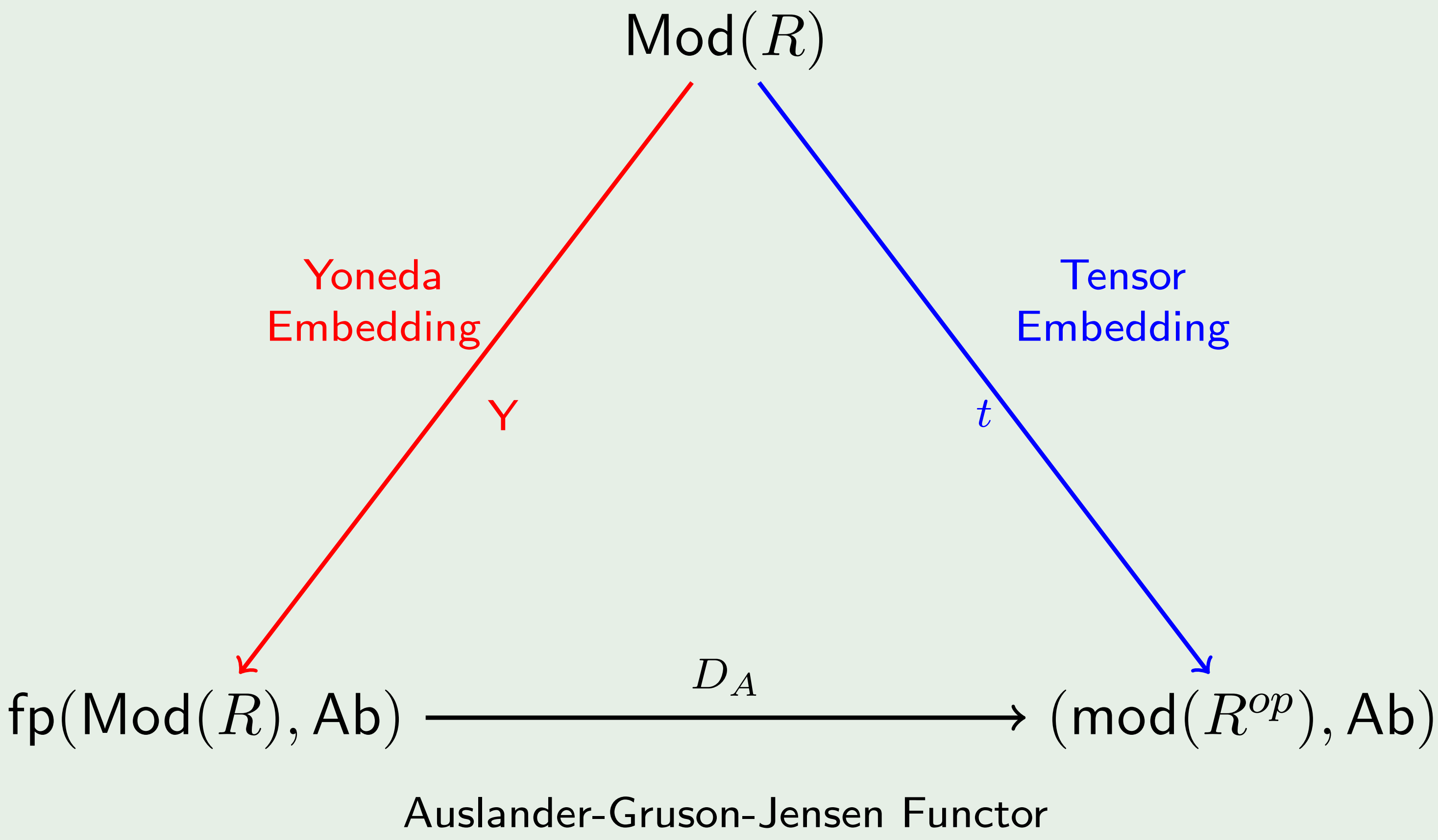
Theorem

The Auslander-Gruson-Jensen functor D_A admits a fully faithful adjoint D^R

$$\mathrm{fp}(\mathrm{Mod}(R), \mathrm{Ab}) \underset{D^R}{\overset{D_A}{\rightleftarrows}} (\mathrm{mod}(R^{op}), \mathrm{Ab})$$

Hence the category $(\mathrm{mod}(R^{op}), \mathrm{Ab})$ embeds into the category of finitely presented functors $\mathrm{fp}(\mathrm{Mod}(R), \mathrm{Ab})$.

Simple Goal: Complete the Diagram of Adjunctions



Notation

Ab - category of abelian groups.

R - ring.

Mod(R) - category of right modules.

mod(R) - category of finitely presented right modules.

Left modules are right modules over R^{op}.

$(\text{mod}(R^{op}), \text{Ab})$ - category of all additive covariant functors.

“Functors”

Convention Concerning Functors

- The word functor will always mean additive functor.
- The morphisms in all functor categories considered will be the natural transformations between functors.
- The representable functors

$$\mathsf{Hom}_R(X, \_\,)\colon \mathsf{Mod}(R) \longrightarrow \mathsf{Ab}$$

  will be abbreviated by $(X, \_\,)$.
- If the word contravariant is not used, the functor is assumed to be covariant.

Yoneda's Lemma

Lemma (Yoneda)

For any $X \in \mathsf{Mod}(R)$ and any functor $F\colon \mathsf{Mod}(R) \to \mathsf{Ab}$,

$$\mathsf{Nat}\big((X, \_\ ), F\big) \cong F(X)$$

1. The isomorphism is defined by $\alpha \mapsto \alpha_X(1_X)$.
2. The isomorphism is natural in both X and F.

Finitely Presented Functors

Definition (Auslander)

A functor $F\colon \mathsf{Mod}(R) \to \mathsf{Ab}$ is called **finitely presented** if there exists a sequence of natural transformations

$$(Y, \_\,) \to (X, \_\,) \to F \to 0$$

such that for any $A \in \mathsf{Mod}(R)$, the sequence of abelian groups

$$(Y, A) \to (X, A) \to F(A) \to 0$$

is exact.

Notation:

$\mathsf{fp}(\mathsf{Mod}(R), \mathsf{Ab}) =$ category of finitely presented functors.

Theorem (Auslander)

fp(Mod(R), Ab) is an abelian category with enough projectives.

Exactness :

$F \to G \to H$ exact in fp(Mod(R), Ab)	$F(A) \to G(A) \to H(A)$ exact in Ab $\forall_A$

Projectives: The projectives are the representable functors.

Resolutions: Any finitely presented functor has a projective resolution of the form:

$$0 \to (Z, \_\ ) \to (Y, \_\ ) \to (X, \_\ ) \to F \to 0$$

Injectives in $\text{fp}(\text{Mod}(R), \text{Ab})$

Proposition (Auslander)

A functor F in $\text{fp}(\text{Mod}(R), \text{Ab})$ is injective if and only if F is right exact.

Theorem (Gentle)

$\text{fp}(\text{Mod}(R), \text{Ab})$ has enough injectives. The injectives are precisely the functors F for which there exists projective modules P, Q and an exact sequence

$$(Q, \_\_) \to (P, \_\_) \to F \to 0$$

The Yoneda Embedding

The functor

$$\mathsf{Y}\colon \mathsf{Mod}(R) \longrightarrow \mathsf{fp}(\mathsf{Mod}(R), \mathsf{Ab})$$

given by

$$\mathsf{Y}(X) = (X, \_\ )$$

is a contravariant embedding, i.e. fully-faithful.

This is called the **Yoneda Embedding**.

Construction of $w\colon \mathrm{fp}(\mathrm{Mod}(R), \mathrm{Ab}) \to \mathrm{Mod}(R)$

Auslander constructed a contravariant functor

$$w\colon \mathrm{fp}(\mathrm{Mod}(R), \mathrm{Ab}) \to \mathrm{Mod}(R)$$

as follows

Step 1: Start with presentation

$$(Y, \_\,) \to (X, \_\,) \to F \to 0$$

Step 2: By Yoneda's lemma $(Y, \_\,) \to (X, \_\,)$ comes from a unique morphism

$$X \to Y$$

Step 3: The exact sequence

$$0 \to w(F) \to X \to Y$$

completely determines w.

Properties of w

Proposition (Auslander)

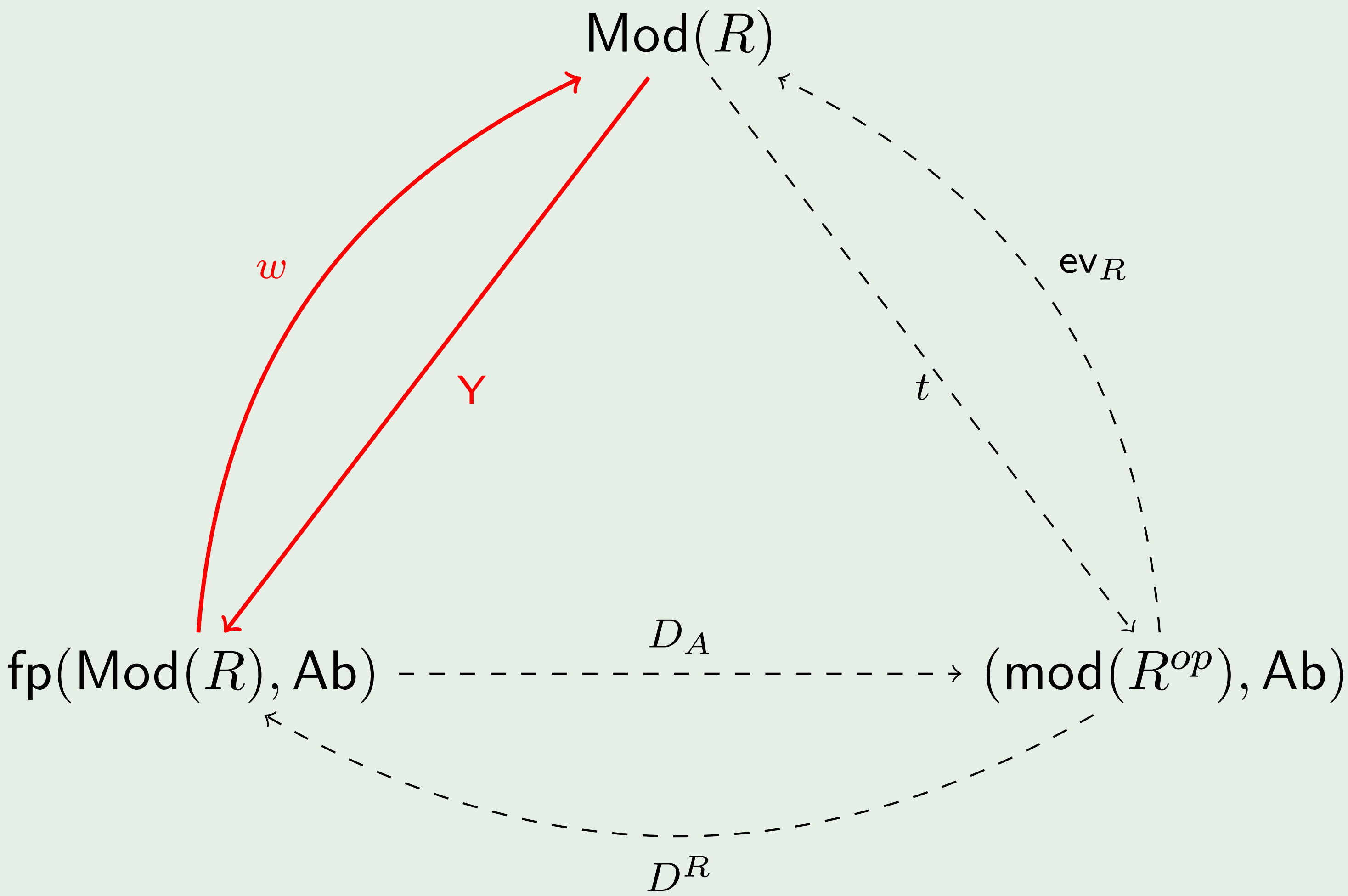
1. w does not depend on the choice of presentation.
2. w is exact.
3. $w(X, \_\ ) = X$.
4. w is an adjoint to the Yoneda embedding

$$\mathsf{Y}\colon \mathsf{Mod}(R) \longrightarrow \mathsf{fp}(\mathsf{Mod}(R), \mathsf{Ab})$$

Remark

Property 4 is not explicitly stated by Auslander but his discussion regarding w implies the result immediately.

First Adjunction



$(\text{mod}(R^{op}), \text{Ab})$

Theorem (?)

$(\text{mod}(R^{op}), \text{Ab})$ is abelian and in fact Grothendieck. Hence it has enough injectives and in fact has injective envelopes.

Exactness :

$F \to G \to H$	$F(A) \to G(A) \to H(A)$
exact in $(\text{mod}(R^{op}), \text{Ab})$	exact in Ab $\quad \forall_A$

Pure Exact Sequences

Definition

A **pure exact sequence** is a short exact sequence of modules

$$0 \to A \to B \to C \to 0$$

such that

$$0 \to A \otimes \_ \to B \otimes \_ \to C \otimes \_ \to 0$$

is exact in $(\text{mod}(R^{op}), \text{Ab})$.

Pure Injectives

Definition

A **pure injective** module M is any module which is injective with respect to pure exact sequences. That is given any pure exact sequence

$$0 \to A \to B \to C \to 0$$

the sequence

$$0 \to (C, M) \to (B, M) \to (A, M) \to 0$$

is exact.

Injectives in $(\text{mod}(R^{op}), \text{Ab})$

Theorem (Gruson, Jensen)

The injectives of $(\text{mod}(R^{op}), \text{Ab})$ are precisely the functors of the form $M \otimes \_$ where M is pure injective.

Tensor Embedding

Theorem (Gruson, Jensen, Lenzing)

The covariant functor $t \colon \mathsf{Mod}(R) \longrightarrow (\mathsf{mod}(R^{op}), \mathsf{Ab})$ given by

$$t(M) = M \otimes \_$$

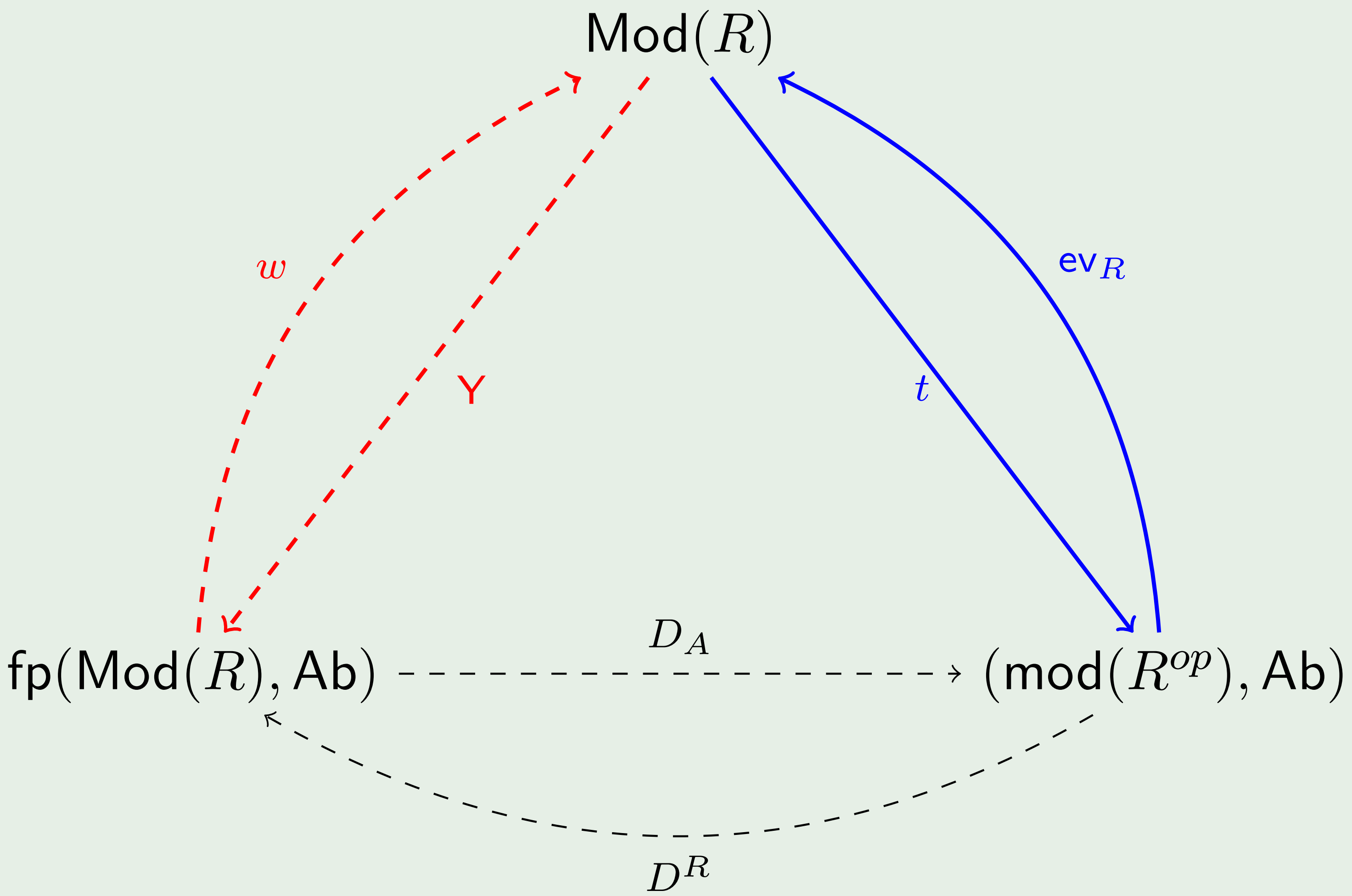
is an embedding, i.e fully faithful. Moreover, the functor t is the left adjoint to evaluation at the ring

$$\mathsf{ev}_R \colon (\mathsf{mod}(R^{op}), \mathsf{Ab}) \longrightarrow \mathsf{Mod}(R)$$

Remark

As a result, given $\alpha \colon M \otimes \_ \to N \otimes \_$ there is some $f \colon M \to N$ such that $\alpha = f \otimes \_$.

Second Adjunction



Finitely Presented Tensor Functors

Theorem (Auslander)

Let V be any left module. Then

$$\_ \otimes V : \mathsf{Mod}(R) \longrightarrow \mathsf{Ab}$$

is finitely presented if and only if V is finitely presented.

Proposition (Auslander)

The tensor functors $\_ \otimes V$ for which $V \in \mathsf{mod}(R^{op})$ are injectives in $\mathsf{fp}(\mathsf{Mod}(R), \mathsf{Ab})$.

The Auslander-Gruson-Jensen Functor

Definition (Auslander-Gruson-Jensen)

Given any finitely presented functor

$$F\colon \mathsf{Mod}(R) \longrightarrow \mathsf{Ab}$$

the functor

$$D_A F\colon \mathsf{mod}(R^{op}) \longrightarrow \mathsf{Ab}$$

is defined by

$$D_A F(L) = \mathsf{Nat}(F, \_ \otimes L)$$

The Auslander-Gruson-Jensen Functor

Theorem (Auslander-Gruson-Jensen)

The above determines an exact contravariant functor

$$D_A \colon \mathsf{fp}(\mathsf{Mod}(R), \mathsf{Ab}) \longrightarrow (\mathsf{mod}(R^{op}), \mathsf{Ab})$$

satisfying

1. $D_A(X, \_\ ) \cong X \otimes \ \_$
2. $D_A(\ \_ \ \otimes V) \cong (V, \ \_\ )$ for finitely presented V.

D^R

Definition

Define

$$D^R = L^0(\mathsf{Y} \circ \mathsf{ev}_R)$$

That is, given $F\colon \mathsf{mod}(R^{op}) \longrightarrow \mathsf{Ab}$ take any injective copresentation

$$0 \longrightarrow F \longrightarrow M \otimes \_ \longrightarrow N \otimes \_$$

$D^R F$ is completely determined by the exact sequence

$$(N, \_) \longrightarrow (M, \_) \longrightarrow D^R F \longrightarrow 0$$

$D_A D^R \cong 1$

Step 1: Start with any injective copresentation

$$0 \longrightarrow F \longrightarrow M \otimes \_ \xrightarrow{f \otimes \_} N \otimes \_$$

Step 2: Applying D^R yields exact sequence:

$$(N, \_) \xrightarrow{(f, \_)} (M, \_) \longrightarrow D^R F \longrightarrow 0$$

Step 3: Apply the exact functor D_A yields exact sequence:

$$0 \longrightarrow D_A D^R F \longrightarrow M \otimes \_ \xrightarrow{f \otimes \_} N \otimes \_$$

D^R is full

Take any two functors $F, G \in (\text{mod}(R^{op}), \text{Ab})$ and injective copresentations

$$0 \longrightarrow F \longrightarrow M \otimes \_ \xrightarrow{f \otimes \_} N \otimes \_$$

$$0 \longrightarrow G \longrightarrow U \otimes \_ \xrightarrow{g \otimes \_} V \otimes \_$$

where M, N, U, V are all pure injective.

D^R is full

Applying D^R yields the following pair of projective presentations

$$
\begin{array}{ccccccc}
(V, \_\,) & \xrightarrow{(g,\ \_\,)} & (U, \_\,) & \longrightarrow & D^R G & \longrightarrow & 0 \\
\downarrow{\scriptstyle (k,\ \_\,)} & & \downarrow{\scriptstyle (h,\ \_\,)} & & \downarrow{\scriptstyle \alpha} & & \\
(N, \_\,) & \xrightarrow{(f,\ \_\,)} & (M, \_\,) & \longrightarrow & D^R F & \longrightarrow & 0
\end{array}
$$

D^R is full

There corresponds in $(\mathsf{mod}(R^{op}), \mathsf{Ab})$:

$$
\begin{array}{ccccccc}
0 & \longrightarrow & F & \longrightarrow & M \otimes \_ & \xrightarrow{f \otimes \_} & N \otimes \_ \\
 & & \Big\downarrow{\scriptstyle \beta} & & \Big\downarrow{\scriptstyle h \otimes \_} & & \Big\downarrow{\scriptstyle k \otimes \_} \\
0 & \longrightarrow & G & \longrightarrow & U \otimes \_ & \xrightarrow{g \otimes \_} & V \otimes \_
\end{array}
$$

D^R is full

Apply D^R

$$
\begin{array}{ccccccc}
(V, \_\_) & \xrightarrow{(g,\ \_\_)} & (U, \_\_) & \longrightarrow & D^R G & \longrightarrow & 0 \\
\downarrow{\scriptstyle (k,\ \_\_)} & & \downarrow{\scriptstyle (h,\ \_\_)} & & \downarrow{\scriptstyle D^R(\beta) = \alpha} & & \\
(N, \_\_) & \xrightarrow{(f,\ \_\_)} & (M, \_\_) & \longrightarrow & D^R F & \longrightarrow & 0
\end{array}
$$

Embedding $(\mathsf{mod}(R), \mathsf{Ab})$ into $\mathsf{fp}(\mathsf{Mod}(R), \mathsf{Ab})$

Proposition

The functor D^R is fully faithful.

What about $D^R D_A$?

Maybe $D^R D_A \cong 1$?

Proposition

$\mathrm{Ker}(D_A) =$ all fp functors arising from pure exact sequences

Meaning of this?

In other words, $D_A F = 0$ if and only if there exists a pure exact sequence

$$0 \to X \to Y \to Z \to 0$$

and projective resolution

$$0 \to (Z, \_\ ) \to (Y, \_\ ) \to (X, \_\ ) \to F \to 0$$

<u>D_A is not faithful</u>.

Relationship between $\mathrm{Im}(D^R)$ and $\mathrm{Ker}(D_A)$

Lemma

For any $G \in (\mathrm{mod}(R^{op}), \mathrm{Ab})$ and any $F \in \mathrm{Ker}(D_A)$

$$\mathrm{Nat}(D^R G, F) = 0$$

Auslander's Blueprint

Theorem (Auslander)

For any finitely presented functor $F\colon \mathsf{Mod}(R) \to \mathsf{Ab}$ there exists an exact sequence

$$0 \longrightarrow F_0 \longrightarrow F \xrightarrow{\varphi} \mathsf{Y}w(F) \longrightarrow F_1 \longrightarrow 0$$

where $w(F_0) \cong w(F_1) \cong 0$. This sequence is functorial in F.

Following Auslander's Blueprint

Theorem

For any finitely presented functor $F\colon \mathsf{Mod}(R) \to \mathsf{Ab}$ there is an exact sequence

$$0 \longrightarrow F_p \longrightarrow D^R D_A F \xrightarrow{\ \gamma\ } F \longrightarrow F^p \longrightarrow 0$$

where $D_A(F_p) \cong D_A(F^p) \cong 0$, and hence arise from pure exact sequences. This sequence is functorial in F.

Constructing γ:

For any $X \in \mathsf{Mod}(R)$ from the injective copresentation

$$0 \longrightarrow X \otimes \_ \longrightarrow M \otimes \_ \longrightarrow N \otimes \_$$

This yields a commutative diagram

$$\begin{array}{ccccccc}
(N, _) & \longrightarrow & (M, _) & \longrightarrow & D^R D_A(X, _) & \longrightarrow & 0 \\
\downarrow 1 & & \downarrow 1 & & \downarrow \gamma_{(X, _)} & & \\
(N, _) & \longrightarrow & (M, _) & \longrightarrow & (X, _) & &
\end{array}$$

Constructing γ:

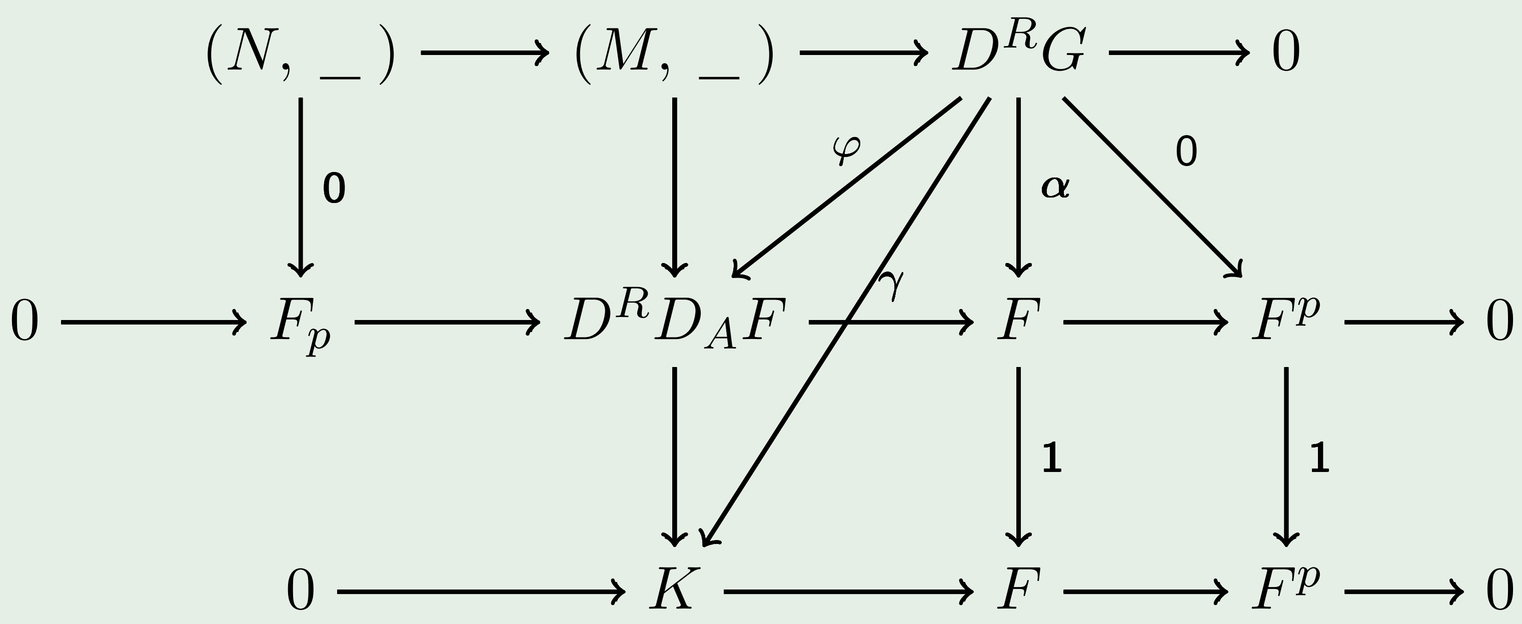
For any $F \in \mathsf{fp}(\mathsf{Mod}(R), \mathsf{Ab})$ and presentation

$$(Y, \_\_) \to (X, \_\_) \to F \to 0$$

$$\begin{array}{ccccccc} D^R D_A(Y, \_\_) & \longrightarrow & D^R D_A(X, \_\_) & \longrightarrow & D^R D_A F & \longrightarrow & 0 \\ \downarrow \gamma_{(Y, \_\_)} & & \downarrow \gamma_{(X, \_\_)} & & \downarrow \gamma_F & & \\ (Y, \_\_) & \longrightarrow & (X, \_\_) & \longrightarrow & F & \longrightarrow & 0 \end{array}$$

The universal property of γ

Da Diagram Chasssse



The Adjunction

Theorem

For any $G \in (\mathsf{mod}(R^{op}), \mathsf{Ab})$ and any $F \in \mathsf{fp}(\mathsf{Mod}(R), \mathsf{Ab})$,

$$\mathsf{Nat}(D^R G, F) \cong \mathsf{Nat}(D_A F, G)$$

which is natural in F and G.

In other words D_A and D^R form an adjoint pair.

Main Result

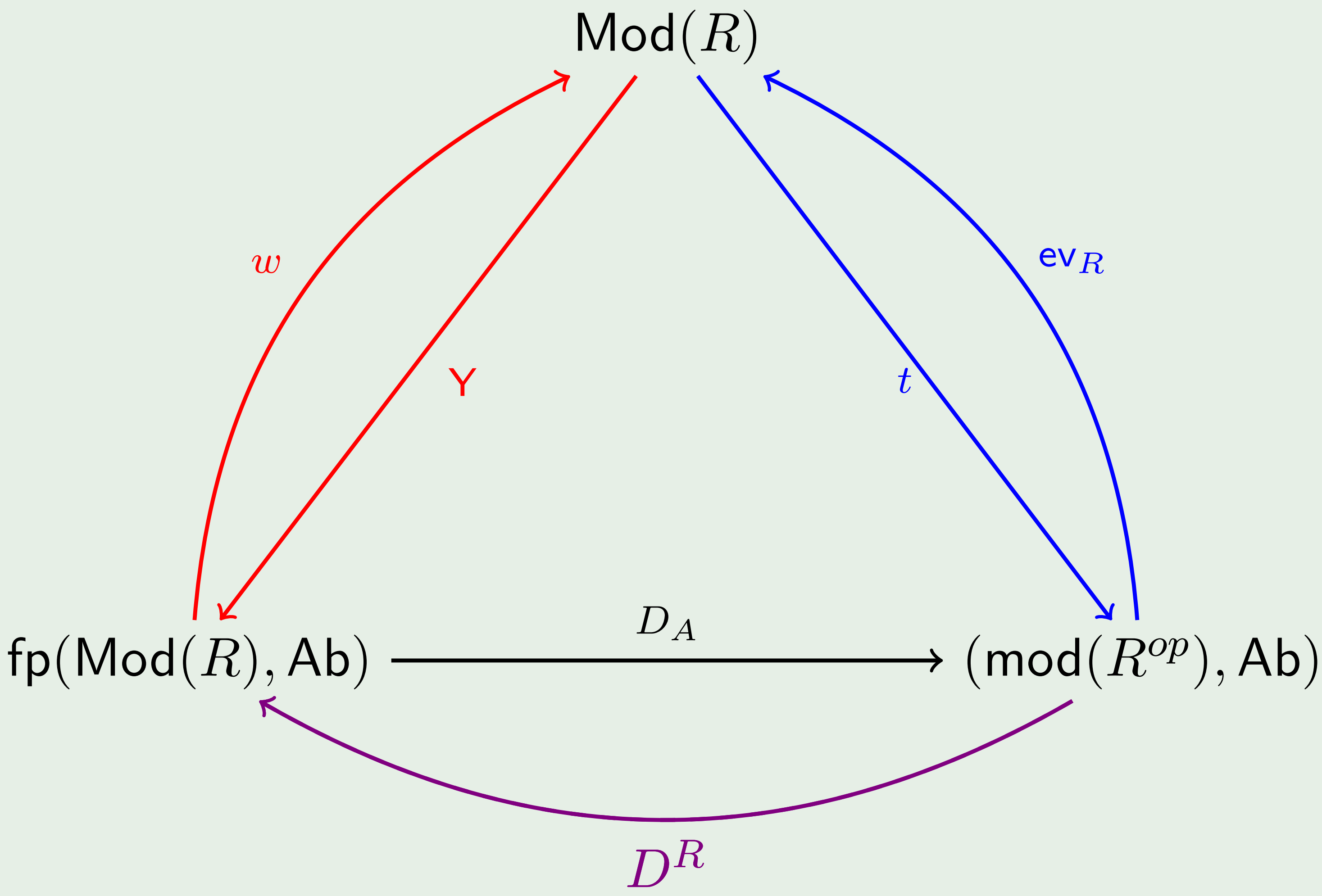
The Auslander-Gruson-Jensen functor D_A admits a fully faithful adjoint D^R

Theorem

$$\mathrm{fp}(\mathrm{Mod}(R), \mathrm{Ab}) \underset{D^R}{\overset{D_A}{\rightleftarrows}} (\mathrm{mod}(R^{op}), \mathrm{Ab})$$

Hence the category $(\mathrm{mod}(R^{op}), \mathrm{Ab})$ embeds into the category of finitely presented functors $\mathrm{fp}(\mathrm{Mod}(R), \mathrm{Ab})$.

The Diagram of Adjunctions



Equivalence of Categories

Corollary

The functors D^R and D_A induce an equivalence of categories

$$\frac{\mathsf{fp}(\mathsf{Mod}(R), \mathsf{Ab})}{\{F \mid D_A F = 0\}} \cong (\mathsf{mod}(R^{op}), \mathsf{Ab})^{op}$$

Universal Property of D_A

$$
\begin{array}{ccc}
\mathrm{fp}(\mathrm{Mod}(R), \mathrm{Ab}) & \xrightarrow{\;D_A\;} & (\mathrm{mod}(R^{op}), \mathrm{Ab}) \\
\Big\downarrow {\scriptstyle \forall E} & \swarrow {\scriptstyle \exists ! \Psi = ED^R} & \\
\mathcal{A} & &
\end{array}
$$

E - exact (covariant), vanishes on $\mathrm{Ker}(D_A)$

$\mathcal{A}$ - abelian.

Ψ - exact contravariant.

Thank You!