Triangular Bases of Quantum Cluster Algebras and Monoidal Categorification

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Outline

Cluster algebras

Cluster algebras are combinatorial objects.

Cluster algebra: $\mathbb{Z}$-subalgebra of a Laurent polynomial ring

- Cluster variables = generators defined recursively by mutations
- Seeds (local charts) = collections of generators + matrices
- Cluster monomials = monomials of cluster variables in the same seeds

Invented by [Fomin-Zelevinsky, 2000] as an combinatorial approach to the dual canonical basis of quantum groups in the sense of Lusztig and Kashiwara.

slow progress for many years

Cluster algebras

Cluster algebras' appearance.

Fruitful in many other areas:

- Combinatorics
- Representation theory of finite dimensional algebras, 2-Calabi-Yau categories
- Higher Teichmüller theory [Fock-Goncharov]
- Poisson geometry [Gekhtman-Shapiro-Vainstein]

Cluster algebras

Cluster algebras' appearance.

- Discrete dynamical systems:
  - [Francesco-Kedem] [FZ] [Inoue-Iyama-Kuniba-Nakanishi-Suzuki]
  - proof of the periodicity conjecture of Y-system [Keller]
- commutative/non-commutative algebraic geometry:
  - Bridgeland's stability conditions of 3-Calabi-Yau categories,
  - Donaldson-Thomas invariants [Kontsevich-Soibelman], Tropical geometry [Gross-Hacking-Keel-Kontsevich]

Monoidal Categorification Conjectures

Read cluster algebras from monoidal categories

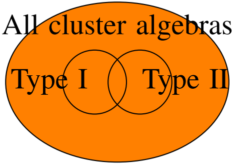
[Hernandez-Leclerc, 09] proposed the monoidal categorification approach to a cluster algebra $\mathscr{A}$:

$\mathscr{A}$		$\mathscr{C}$	Monoidal cateogry
$+$		$\oplus$	
$\cdot$		$\otimes$	
$\mathscr{A}$	$\simeq$	$K_0(\mathscr{C})$	Grohendieck ring
cluster monomials	$\subset$	simple objects	
good basis	$=$	{simples}	

- Find the monoidal category such that $\mathscr{A} \simeq K_0(\mathscr{C})$?
- The cluster monomials are simples?

Monoidal Categorification Conjectures

World of cluster algebras



After quantization, the quantum cluster algebras $\mathscr{A}_q$ are related to,

- in Type I, the quantum groups of symmetric Cartan type:
  $\mathscr{A}_q \simeq K_0(\mathrm{KLR} - alg\ f.d.\ mod)$ $(\sim\ U_q(\mathfrak{n})^{*,gr})$
  [Geiss-Leclerc-Schröer];[Khovanov-Lauda] [Rouquier]
- in Type II, the quantum affine algebras of type ADE:
  $\mathscr{A}_q \simeq K_t(U_q(\hat{\mathfrak{g}})\ f.d.\ mod)$ t-deformed Grothendieck ring
  [Hernandez-Leclerc];[Varagnolo-Vasserot] [Nakajima][H.]

Monoidal Categorification Conjectures

Conjectures and results

Monoidal Categorification Conjecture

The cluster monomials are simples?

Theorem ([Lampe] [Hernandez-Leclerc] [Nakajima] [Kimura-Q.])

Partial results for type I and type II.

Theorem (Q., 15)

For all type II and some type I (adaptable word): The monoidal categorification conjecture is true. The Fock-Goncharov conjecture is also true.

Theorem (Kang-Kashiwara-Kim-Oh, 15)

For all type I: The monoidal categorification conjecture is true.

A rank 2 example

Example (Quantum cluster variables)

Take matrices $B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $\Lambda = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

Initial cluster variable: X_1, X_2.

Quantum torus $\mathscr{T}$: Laurent polynomial ring $(\mathbb{Z}[q^{\pm\frac{1}{2}}][X_1^{\pm}, X_2^{\pm}], +, \cdot)$

q-twisted product $X^g * X^h = q^{\frac{1}{2}g\Lambda h^T} X^{g+h}$,

bar involution $\overline{q^s X^g} = q^{-s} X^g$.

The quantum cluster variables $\{X_k\}$ by mutations:

$$X_k * X_{k+2} = q^{-\frac{1}{2}} X_{k+1} + 1, \ \forall k \in \mathbb{Z}.$$

Example (Quantum cluster algebra of rank 2)

Seeds (local charts): $(\{X_k,\ X_{k+1}\}, (-1)^{k+1}B, (-1)^{k+1}\Lambda)$.

Quantum cluster algebra $\mathscr{A}_q = \mathbb{Z}[q^{\pm\frac{1}{2}}][X_k]_{k\in\mathbb{Z}}$.

A rank 2 example

Example (Previous example)

$$X_3 = X^{(-1,1)} + X^{(-1,0)} (= X_1^{-1} \cdot X_2 + X_1^{-1})$$
$$X_4 = X^{(0,-1)} + X^{(-1,-1)} + X^{(-1,0)}.$$

General cluster algebras

Definition (Berenstein-Zelevinsky, 05)

In general, for any given skew-symmetrizable $m \times n$ matrix B, $m \geq n$, and a compatible skew-symmetric $m \times m$ matrix Λ, we can define the quantum cluster algebra

$$\mathscr{A}_q = \mathscr{A}_q((X_1, \dots, X_m), B, \Lambda).$$

Theorem (Laurent phenomenon [Fomin-Zelevinsky][Berenstein-Zelevinsky])

Any cluster variable is a Laurent polynomial in $\mathbb{Z}[X_1^{\pm}, \dots, X_m^{\pm}]$. *Any quantum cluster variable is a Laurent polynomial in* $\mathbb{Z}[q^{\pm\frac{1}{2}}][X_1^{\pm}, \dots, X_m^{\pm}]$.

Cluster expansions

Theorem ([Derksen-Weyman-Zelevinsky][Plamondon][Nagao][Gross-Hacking-Keel-Kontsevich], +[Tran])

Define $Y_k = X^{(Be_k)^T}$, then any quantum cluster variable is always a Laurent polynomial of the form:

$$X^g(1+\sum_{0\neq v\in\mathbb{N}^n} c_v Y^v),\ c_v\in\mathbb{Z}[q^{\pm\frac{1}{2}}].$$

Example (Previous example)

Recall $B=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$, we have $Y_1=X_2,\ Y_2=X_1^{-1}$,
$X_3=X^{(-1,1)}(1+Y_1),\ X_4=X^{(0,-1)}(1+Y_2+Y_1Y_2)$

Degrees and partial order

Definition

A Laurent polynomial Z of the form $X^g(1+\sum_{0\neq v\in\mathbb{N}^n} c_v Y^v)$ is said to be pointed at the degree g.
In this case, we denote $\deg Z = g$.

Definition

We say degrees $g \geq g'$ if $\deg X^g Y^v = \deg X^{g'}$ for some $v \in \mathbb{N}^n$.

Then the above Z has the unique maximal degree g.

Example (Previous example)

Recall $Y_1 = X_2,\ Y_2 = X_1^{-1}$,
$X_3 = X^{(-1,1)}(1+Y_1),\ X_4 = X^{(0,-1)}(1+Y_2+Y_1Y_2)$
$\deg X_4 > \deg X_3 = \deg X_4 Y_2 Y_1^2$.

Triangular basis for an initial seed

Choose and work in an initial seed $(\{X_1, \dots, X_n, \dots, X_m\}, B, \Lambda)$.
Assume that there exists quantum cluser varaibles I_k, $1 \leq k \leq n$, such that $pr_n \deg I_k = -e_k$
(pr_n = projection to the first n-coordinates).

Definition

The triangular basis $\mathscr{L}$ is the basis of $\mathscr{A}_q$ such that

1. $X_i,\ I_k \in \mathscr{L}$
2. elements of $\mathscr{L}$ are bar-invariant
3. (Parametrization) elements of $\mathscr{L}$ have unique maximal degrees with coefficient 1, such that $\deg : \mathscr{L} \simeq \mathbb{Z}^m$.
4. (Triangularity) $\forall X_i, b_1 \in \mathscr{L}$, there exists some $s \in \frac{\mathbb{Z}}{2}$ such that
$$q^s X_i * b_1 = b_2 + \sum_b a_b \cdot b,$$
where $b_2, b \in \mathscr{L}$, coefficients $a_b \in q^{-\frac{1}{2}}\mathbb{Z}[q^{-\frac{1}{2}}]$, $\deg X_i + \deg b_1 = \deg b_2 > \deg b$.

Common triangular basis and Fock-Goncharov conjecture

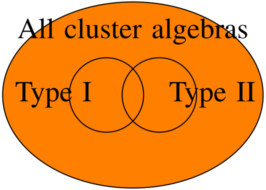
- A triangular basis, if it exists, is unique.
- The notion of the triangular basis depends on the chosen initial seed (local chart).
  In particular, the degree of a basis element will differ when we change the initial seed.

Definition

A basis $\mathscr{L}$ is called the common triangular basis, if it is the triangular basis for any seed and,
moreover, its parametrization in different seeds verifies the Fock-Goncharov conjecture:

$$\begin{array}{ccc} \mathscr{L} & \simeq & \mathbb{Z}^m \\ || & & \updownarrow \quad \text{mutation of tropical } \mathbb{Z}\text{-points} \\ \mathscr{L} & \simeq & \mathbb{Z}^m \end{array} .$$

Main Theorem



Theorem ([Q. 15])

For some of type I (adaptable Coxeter element case) and all type II, the basis of simples produces the common triangular basis of the quantum cluster algebra, which also verifies the Fock-Goncharov conjecture.

Corollary ([Q. 15])

Monoidal categorification conjecture is true in these cases..

Idea of the proof

How to find the common triangular basis?

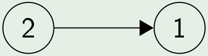
- Try to proceed by induction on seeds:
  Key observation: for a basis with positive structure constants,
  - If it contains all cluster variables (monoidal categorification conjecture), then it has good parametrization (Fock-Goncharov conjecture)
  - If it has good parametrization, then it contains all cluster variables.
- In practice, we do induction on seeds to show that the basis produced by simples satisfy both conjectures.

Cluster algebra associated with a quiver

For any $m \geq n \in \mathbb{N}$ and quiver Q (finite oriented graph) with vertices $\{1, \dots, n, \dots, m\}$, we can define an $m \times n$ matrix $B = (b_{ij})$: $b_{ij} = |\text{arrows } i \to j| - |\text{arrows } j \to i|$.
Therefore, we can associate cluster algebra with any quiver.
We can further impose a quantization if $\mathrm{rk} B = n$.

Example (Previous example)

Choose $m = n = 2$. The following quiver gives us the 2×2 matrix $B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

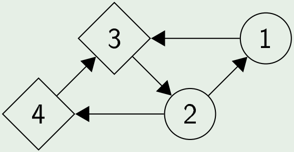


By choosing a special quantization, we get the previous quantum cluster algebra.

Cluster algebra associated with a quiver

Example

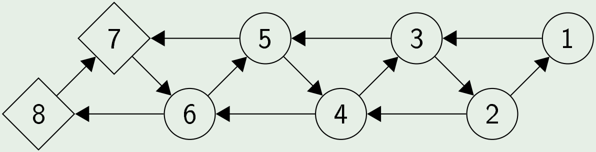
Choose $m = 4,\ n = 2.$ The following quiver gives us the 4×2 matrix $B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \\ -1 & 1 \\ 0 & -1 \end{pmatrix}.$



Cluster algebra: Type II

Example (Type II: $U_q(\hat{\mathfrak{sl}}_3)$-mod)

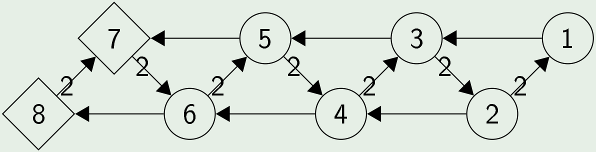
The following quiver arising from a level 3 subcategory of $U_q(\hat{\mathfrak{sl}}_3)$-mod.



Cluster algebra: Type I adaptable word

Example (Type I: adaptable word)

The following quiver is associated with the adaptable word $\underline{i} = (2,1,2,1,2,1,2,1)$ and the Cartan matrix $C = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$.



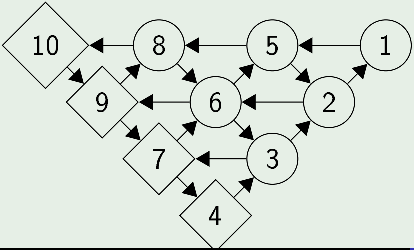
Cluster algebra: Type I and Type II

Example (Type I and type II)

The following quiver is associated with the adaptable word $\underline{i} = (1,2,1,3,2,1,4,3,2,1)$ and the Cartan matrix

$$C = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}.$$

It also arises from a subcategory of $U_\varepsilon(\hat{\mathfrak{sl}}_5)$-mod.



Cluster algebra: Type I non-adaptable word

Example (Type I: non-adaptable word)

The following quiver is associated with the non-adaptable word $\underline{i} = (2,3,2,1,2,1,3,1,2,1)$ and the Cartan matrix

$$C = \begin{pmatrix} 2 & -3 & -2 \\ -3 & 2 & -2 \\ -2 & -2 & 2 \end{pmatrix}.$$

It is not included in our Theorem.