Representation homology and derived character maps

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1. Classical representation schemes.
2. Derived representation schemes and representation homology.
3. Derived character maps.
Assumption: $k$ is a fixed field of $\text{char}(k) = 0$, all algebras are over $k$, $\otimes$ denotes $\otimes_k$. A graded algebra $B$ is commutative if for $a, b \in B$

$$ab = (-1)^{\deg(a)\deg(b)}ba$$

Let $A \in \text{Alg}_k$ be an associative algebra, $V = k^n$ an $n$-dimensional vector space.

By $\text{Rep}_n(A)$ we denote the moduli space of representations of $A$ in $k^n$.

\textbf{Example.} $\text{Rep}_n(k\langle x_1, \ldots, x_r \rangle) = \text{Mat}_n^{\times r} \cong A^{rn^2}$.

\textbf{Example.} $\text{Rep}_n(k[x_1, \ldots, x_r]) \subset \text{Mat}_n^{\times r}$ is the closed subscheme, consisting of tuples $(B_1, \ldots, B_r)$ of pair-wise commuting matrices.
Characters define a linear map $\text{Tr}: A \to k[\text{Rep}_n(A)]$

$$a \mapsto [\text{Tr}(a): \rho \mapsto \text{tr}(\rho(a))] , \forall \rho \in \text{Rep}_n(A)$$

This map factors as

$$\begin{array}{ccc}
A & \xrightarrow{\text{Tr}} & k[\text{Rep}_n(A)] \\
\downarrow & & \uparrow_{i}
\end{array}$$

$$A/[A, A] \longrightarrow k[\text{Rep}_n(A)]^{\text{GL}_n}$$

The map $A/[A, A] \to k[\text{Rep}_n(A)]^{\text{GL}_n}$ will be called the character map.

**Theorem (Procesi)**

*The induced map* $\text{Sym}(\text{Tr}): \text{Sym}(A/[A, A]) \to k[\text{Rep}_n(A)]^{\text{GL}_n}$ *is surjective.*
In general, $\text{Rep}_n(A)$ is “badly behaved,” for example, it is quite singular even for “nice” algebras (e.g. $A = k[x_1, \ldots, x_d], d > 1$)

**Solution:** “resolve singularities” by deriving $\text{Rep}_n$.

Call the functor $(-)_n : \text{Alg}_k \to \text{ComAlg}_k$ sending

$$A \mapsto A_n := k[\text{Rep}_n(A)]$$

the **representation functor**.

It extends naturally to $(-)_n : \text{DGA}_k \to \text{CDGA}_k$.

**Problem:** The functor $(-)_n$ is not “exact”, i.e. it does not preserve quasi-isomorphisms.
Theorem (Berest–Khachatryan–Ramadoss)

The functor $(-)_n$ has a total left derived functor $\mathbb{L}(-)_n$ computed by $\mathbb{L}(A)_n = R_n$ for any resolution $R \sim A$. The algebra $\mathbb{L}(A)_n$ does not depend on the choice of resolution, up to quasi-isomorphism.

For $A \in \text{Alg}_k$, a resolution is any semi-free DG algebra $R \in \text{DGA}_k$ with a surjective quasi-isomorphism $R \sim A$.

Denote $\mathbb{L}A_n$ by $\text{DRep}_n(A)$, call it derived representation scheme.

Example: If $A = k[x, y]$, take $R = k\langle x, y, \lambda \rangle$ with $\deg(x) = \deg(y) = 0$, $\deg(\lambda) = 1$ and $d\lambda = xy - yx$.

Then $\text{DRep}_n(A) = k[x_{ij}, y_{ij}, \lambda_{ij}]$ with $\deg(\lambda_{ij}) = 1$ and

$$d\lambda_{ij} = \sum_{k=1}^{n} x_{ik} y_{kj} - y_{ik} x_{kj}$$
Define \( n \)-dimensional **representation homology** by

\[
H_\bullet(A, n) := H_\bullet[\text{DRep}_n(A)]
\]

**Facts:**

1. \( H_0(A, n) \cong k[\text{Rep}_n(A)] =: A_n \).
2. If \( \text{Rep}_n(A) = \emptyset \), then \( H_\bullet(A, n) = 0 \).
3. For \( A \) formally smooth, \( H_p(A, n) = 0 \) for \( \forall n \geq 1 \) and \( p \geq 1 \).
4. \( \text{DRep}_1(A) \cong R_{ab} \) for any resolution \( R \twoheadrightarrow A \), so

\[
H_\bullet(A, 1) \cong H_\bullet(R_{ab})
\]
Example: polynomial algebra on two variables

Let $A = k[x, y]$, $R = k\langle x, y, \lambda \rangle$ with $d\lambda = xy - yx$. Then $\text{DRep}_1(A) \simeq k[x, y, \lambda]$ with zero differential, so

$$H_\bullet(k[x, y], 1) \simeq k[x, y] \oplus k[x, y].\lambda$$

$$\text{deg}=0 \quad \text{deg}=1$$

$$H_\bullet(k[x, y], 2) \simeq k[x, y]_2 \otimes \text{Sym}(\xi, \tau, \eta)/I$$

with $\xi, \tau, \eta$ of degree 1 and $I$ the ideal generated by the relations

- $x_{12}\eta - y_{12}\xi = (x_{12}y_{11} - y_{12}x_{11})\tau$
- $x_{21}\eta - y_{21}\xi = (x_{21}y_{22} - y_{21}x_{22})\tau$
- $(x_{11} - x_{22})\eta - (y_{11} - y_{22})\xi = (x_{11}y_{22} - y_{11}x_{22})\tau$
- $\xi\eta = y_{11}\xi\tau - x_{11}\eta\tau = y_{22}\xi\tau - x_{22}\eta\tau$

**Theorem (Berest-Felder-Ramadoss)**

*For $i > n$ we have $H_i(k[x, y], n) = 0$.***
Example: q-polynomials and dual numbers

Let $q \in k^\times$, and define $k_q[x, y] = k\langle x, y \rangle/(xy = qyx)$.

**Theorem (Berest–Felder–Ramadoss)**

*If $q$ is not a root of 1, then for all $n \geq 1$*

$$H_p(k_q[x, y], n) = 0, \quad \forall p > 0$$

For $A = k[x]/(x^2)$ the minimal resolution is $R = k\langle t_0, t_1, t_2, \ldots \rangle$ with $\deg t_i = i$ and

$$dt_p = t_0 t_{p-1} - t_1 t_{p-2} + \cdots + (-1)^{p-1} t_{p-1} t_0$$

In this case even for $H_\bullet(A, 1) = H_\bullet(R_{ab})$ don't have a good description.
Relation to Lie homology

Let $C$ be a (augmented) DG coalgebra Koszul dual to $A \in \text{Alg}_k$ (augmented), i.e. $\Omega(C) \xrightarrow{\sim} A$.

**Theorem (Berest–Felder–P–Ramadoss–Willwacher)**

There is an isomorphism

$$H_\bullet(A, n) \simeq H_\bullet(\mathfrak{gl}_n^*(\tilde{C}); k), \quad H_\bullet(A, n)^{\text{GL}_n} \simeq H_\bullet(\mathfrak{gl}_n^*(C), \mathfrak{gl}_n^*(k); k)$$

If $\dim(C) < \infty$, take $E = C^*$ the linear dual DG algebra. Then

$$H_\bullet(A, n) \simeq H^{-\bullet}(\mathfrak{gl}_n(\tilde{E}); k), \quad H_\bullet(A, n)^{\text{GL}_n} \simeq H^{-\bullet}(\mathfrak{gl}_n(E), \mathfrak{gl}_n(k); k)$$
**Want:** relate $H_\bullet(A, n)$ to more computable invariants.

**Proposition (Berest-Khachatryan-Ramadoss)**

For any algebra $A \in \text{Alg}_k$ and any $n$ there exists a **canonical** derived character map

$$\text{Tr}_n(A)_\bullet : HC_\bullet(A) \to H_\bullet(A, n)^{GL_n},$$

extending the original character map

$$\text{Tr}: HC_0(A) = A/[A, A] \to A_n^{GL_n}$$
**Goal:** compute derived character maps for $A = \text{Sym}(W)$.

For simplicity, **assume $n = 1$** (i.e. only consider $H_i(A, 1)$).

$\text{Tr}(A)_i$ factors through the *reduced* cyclic homology $\overline{HC}_i(A)$.

For $A = \text{Sym}(W)$,

$$\overline{HC}_i(A) \simeq \Omega^i(W)/d\Omega^{i-1}(W), \quad \Omega^i(W) \simeq \text{Sym}(W) \otimes \Lambda^i(W)$$

Thus, we can think of $\text{Tr}(A)_i$ as maps

$$\text{Tr}(A)_i: \Omega^i(W) \to H_i(A, 1)$$
Example: $A = k[x, y]$

- $\text{DRep}_1(k[x, y]) \cong k[x, y, \lambda]$ with zero differential.
- The character $\text{Tr}_0: k[x, y] \rightarrow k[x, y, \lambda]$ is given by
  \[
  \text{Tr}_0(P(x, y)) = P(x, y)
  \]
  for any $P(x, y) \in k[x, y]$.
- The character $\text{Tr}_1: \Omega^1(A) \rightarrow k[x, y, \lambda]$ is given by
  \[
  \text{Tr}_1(P(x, y)dx + Q(x, y)dy) = \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \lambda
  \]
$\text{Tr}(A)_1$ for $A = \text{Sym}(W)$

For $A = \text{Sym}(W) \simeq k[x_1, \ldots, x_m]$

$$\text{DRep}_1(A) \simeq \text{Sym}(W) \otimes \text{Sym} \left( \underbrace{\Lambda^2(W) \oplus \cdots \oplus \Lambda^m(W)}_{\text{deg}=1 \oplus \cdots \oplus \text{deg}=m-1} \right).$$

with **zero differential**, so $H_\bullet(A, 1) \simeq \text{DRep}_1(A)$.

$$\lambda(v_1, v_2, \ldots, v_p) := v_1 \wedge v_2 \wedge \ldots \wedge v_p \in \underbrace{\Lambda^p(W)}_{\text{deg}=p-1} \subset \text{DRep}_1(A)$$

**Proposition**

For $\alpha = \sum P_i dx_i \in \Omega^1(A)$ the map $\text{Tr}(A)_1$ is given by

$$\text{Tr}(A)_1(\alpha) = \sum_{i<j} \left( \frac{\partial P_i}{\partial x_j} - \frac{\partial P_j}{\partial x_i} \right) \lambda(x_i, x_j) \in H_\bullet(A, 1)$$
Example: $\text{Tr}_2$ for $A = k[x, y, z]$

Take $\omega = Pdx \wedge dy + Qdy \wedge dz + Rdz \wedge dx \in \Omega^2(A)$. Then $\text{Tr}(A)_2(\omega)$ is given by

$$M\lambda(x, y, z) + M_y\lambda(x, y)\lambda(y, z) + M_z\lambda(y, z)\lambda(z, x) + M_x\lambda(z, x)\lambda(x, y),$$

where

$$M := Pz + Qx + Ry$$

and for a polynomial $F$, $F_q$ denotes $\frac{\partial F}{\partial q}$.

$\text{Tr}(A)_2 = D \circ d_{dR}$, where

$$D = s^{-1} + \tilde{D}: \Omega^3 \to H_\bullet(A, 1)$$

is a certain canonical differential operator on differential forms.
Abstract Chern–Simons forms

Let \( \mathcal{A} \) be a \textbf{cohomologically} graded commutative DG algebra, \( \mathfrak{g} \) a finite dimensional Lie algebra.

A \( \mathfrak{g} \)-valued \textbf{connection} is an element \( \theta \in \mathcal{A}^1 \otimes \mathfrak{g} \).

Its \textbf{curvature} is \( \Theta := d\theta + \frac{1}{2} [\theta, \theta] \), and Bianchi identity holds:

\[
d\Theta = [\Theta, \theta]
\]

If \( P \in k[\mathfrak{g}]^{ad} \), \( \deg(P) = r \), for \( \alpha \in \mathcal{A} \otimes \text{Sym}^r(\mathfrak{g}) \) define \( P(\alpha) \in \mathcal{A} \) via

\[
\mathcal{A} \otimes \text{Sym}^r(\mathfrak{g}) \xrightarrow{\frac{1}{r!} \text{id} \otimes \text{ev}_P} \mathcal{A}
\]

Then \( P(\Theta^r) \in \mathcal{A}^{2r} \) is exact, and there exists \( \text{CS}_P(\theta) \in \mathcal{A}^{2r-1} \) such that \( d \text{CS}_P(\theta) = P(\Theta^r) \) with \( \text{CS}_P(\theta) \) is given explicitly by

\[
\text{CS}_P(\theta) = r! \int_0^1 P(\theta \wedge \Theta_t^{r-1}) dt
\]

where \( \Theta_t = t\Theta + \frac{1}{2}(t^2 - t)[\theta, \theta] \).
Take $\mathcal{A} = \text{hom}(\Omega^\bullet(W), R_{ab})$, $g = k$, and $P_r = x^r \in k[g] \simeq k[x]$.

**Theorem (Berest-Felder-P-Ramadoss-Willwacher)**

There is a canonical $k$-valued connection $\theta$ in $\mathcal{A}$ such that the derived character map $\text{Tr}(A)_\bullet : \Omega^\bullet(A) \to R_{ab} \simeq H_\bullet(A, 1)$ is given by

$$\text{Tr}(A)_\bullet = \sum_{r=0}^{\infty} \text{CS}_{P_r}(\theta) \circ d.$$

Here, $\theta(P(x_1, \ldots, x_m)dx_{i_1} \ldots dx_{i_p}) = P(0, \ldots, 0)x_{i_1, \ldots, x_{i_p}}$

**Remark:** this allows to get explicit formulas for all derived character maps.

