Representation homology and derived character maps

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Plan

1. Classical representation schemes.
2. Derived representation schemes and representation homology.
3. Derived character maps.

Representation schemes

Assumption: k is a fixed field of $char(k) = 0$, all algebras are over k, $\otimes$ denotes $\otimes_k$. A **graded** algebra B is **commutative** if for $a, b \in B$

$$ab = (-1)^{\deg(a)\deg(b)} ba$$

Let $A \in \mathbf{Alg}_k$ be an associative algebra, $V = k^n$ an n-dimensional vector space.

By $\mathrm{Rep}_n(A)$ we denote the moduli space of representations of A in k^n.

Example. $\mathrm{Rep}_n(k\langle x_1, \dots, x_r\rangle) = \mathrm{Mat}_n^{\times r} \simeq \mathbb{A}^{rn^2}$.

Example. $\mathrm{Rep}_n(k[x_1, \dots, x_r]) \subset \mathrm{Mat}_n^{\times r}$ is the closed subscheme, consisting of tuples $(B_1, \dots, B_r)$ of pair-wise commuting matrices.

Character map

Characters define a linear map $\mathsf{Tr}\colon A \to k[\mathrm{Rep}_n(A)]$

$$a \mapsto [\mathsf{Tr}(a)\colon \rho \mapsto \mathsf{tr}(\rho(a))]\,,\ \forall \rho \in \mathrm{Rep}_n(A)$$

This map factors as

$$\begin{array}{ccc} A & \xrightarrow{\mathsf{Tr}} & k[\mathrm{Rep}_n(A)] \\ \downarrow & & \uparrow i \\ A/[A,A] & \longrightarrow & k[\mathrm{Rep}_n(A)]^{\mathsf{GL}_n} \end{array}$$

The map $A/[A,A] \to k[\mathrm{Rep}_n(A)]^{\mathsf{GL}_n}$ will be called the **character map**.

Theorem (Procesi)

The induced map $\mathsf{Sym}(\mathsf{Tr})\colon\ \mathsf{Sym}(A/[A,A]) \to k[\mathrm{Rep}_n(A)]^{\mathsf{GL}_n}$ *is surjective.*

Extension to DG algebras

In general, $\mathrm{Rep}_n(A)$ is "badly behaved," for example, it is quite singular even for "nice" algebras (e.g. $A = k[x_1, ..., x_d], d > 1$)

Solution: "resolve singularities" by **deriving** Rep_n.

Call the functor $(-)_n \colon \mathbf{Alg}_k \to \mathbf{ComAlg}_k$ sending

$$A \mapsto A_n := k[\mathrm{Rep}_n(A)]$$

the **representation functor**.

It extends naturally to $(-)_n \colon \mathbf{DGA}_k \to \mathbf{CDGA}_k$.

Problem: The functor $(-)_n$ is not "exact", i.e. it does not preserve quasi-isomorphisms.

Derived representation functor

Theorem (Berest–Khachatryan–Ramadoss)

The functor $(-)_n$ has a total left derived functor $\mathbb{L}(-)_n$ computed by $\mathbb{L}(A)_n = R_n$ for any resolution $R \overset{\sim}{\twoheadrightarrow} A$. The algebra $\mathbb{L}(A)_n$ **does not depend** *on the choice of resolution, up to quasi-isomorphism.*

For $A \in \mathrm{Alg}_k$, a **resolution** is any semi-free DG algebra $R \in \mathrm{DGA}_k$ with a surjective quasi-isomorphism $R \overset{\sim}{\twoheadrightarrow} A$.

Denote $\mathbb{L}A_n$ by $\mathrm{DRep}_n(A)$, call it **derived representation scheme**.

Example: If $A = k[x, y]$, take $R = k\langle x, y, \lambda\rangle$ with $\deg(x) = \deg(y) = 0$, $\deg(\lambda) = 1$ and $d\lambda = xy - yx$.

Then $\mathrm{DRep}_n(A) = k[x_{ij}, y_{ij}, \lambda_{ij}]$ with $\deg(\lambda_{ij}) = 1$ and

$$d\lambda_{ij} = \sum_{k=1}^{n} x_{ik}y_{kj} - y_{ik}x_{kj}$$

Representation homology

Define n-dimensional **representation homology** by

$$H_\bullet(A,n) := H_\bullet\left[\mathrm{DRep}_n(A)\right]$$

Facts:

1. $H_0(A,n) \simeq k[\mathrm{Rep}_n(A)] =: A_n$.
2. If $\mathrm{Rep}_n(A) = \emptyset$, then $H_\bullet(A,n) = 0$.
3. for A formally smooth, $H_p(A,n) = 0$ for $\forall n \geq 1$ and $p \geq 1$.
4. $\mathrm{DRep}_1(A) \simeq R_{ab}$ for any resolution $R \stackrel{\sim}{\twoheadrightarrow} A$, so

$$H_\bullet(A,1) \simeq H_\bullet(R_{ab})$$

Example: polynomial algebra on two variables

Let $A = k[x, y]$, $R = k\langle x, y, \lambda\rangle$ with $d\lambda = xy - yx$. Then $\mathrm{DRep}_1(A) \simeq k[x, y, \lambda]$ with zero differential, so

$$H_\bullet(k[x, y], 1) \simeq \underbrace{k[x, y]}_{\deg=0} \oplus \underbrace{k[x, y].\lambda}_{\deg=1}$$

$$H_\bullet(k[x, y], 2) \simeq k[x, y]_2 \otimes \mathbf{Sym}(\xi, \tau, \eta)/I$$

with ξ, τ, η of degree 1 and I the ideal generated by the relations

- $x_{12}\eta - y_{12}\xi = (x_{12}y_{11} - y_{12}x_{11})\tau$
- $x_{21}\eta - y_{21}\xi = (x_{21}y_{22} - y_{21}x_{22})\tau$
- $(x_{11} - x_{22})\eta - (y_{11} - y_{22})\xi = (x_{11}y_{22} - y_{11}x_{22})\tau$
- $\xi\eta = y_{11}\xi\tau - x_{11}\eta\tau = y_{22}\xi\tau - x_{22}\eta\tau$

Theorem (Berest-Felder-Ramadoss)

For $i > n$ we have $H_i(k[x, y], n) = 0$.

Example: q-polynomials and dual numbers

Let $q \in k^\times$, and define $k_q[x,y] = k\langle x,y\rangle/(xy = qyx)$.

Theorem (Berest–Felder–Ramadoss)

*If q is **not** a root of 1, then for all $n \geq 1$*

$$H_p(\, k_q[x,y]\,,\, n) = 0, \quad \forall p > 0$$

For $A = k[x]/(x^2)$ the minimal resolution is $R = k\langle t_0, t_1, t_2, \dots\rangle$ with $\deg t_i = i$ and

$$dt_p = t_0 t_{p-1} - t_1 t_{p-2} + \dots + (-1)^{p-1} t_{p-1} t_0$$

In this case even for $H_\bullet(A,1) = H_\bullet(R_{ab})$ don't have a good description.

Relation to Lie homology

Let C be a (augmented) DG coalgebra Koszul dual to $A \in \mathbf{Alg}_k$ (augmented), i.e. $\Omega(C) \xrightarrow{\sim} A$.

Theorem (Berest–Felder–P–Ramadoss–Willwacher)

There is an isomorphism

$$H_\bullet(A, n) \simeq H_\bullet(\mathfrak{gl}_n^*(\bar{C}); k), \quad H_\bullet(A, n)^{\mathrm{GL}_n} \simeq H_\bullet(\mathfrak{gl}_n^*(C), \mathfrak{gl}_n^*(k); k)$$

If $\dim(C) < \infty$, *take* $E = C^*$ *the linear dual DG algebra. Then*

$$H_\bullet(A, n) \simeq H^{-\bullet}(\mathfrak{gl}_n(\bar{E}); k), \quad H_\bullet(A, n)^{\mathrm{GL}_n} \simeq H^{-\bullet}(\mathfrak{gl}_n(E), \mathfrak{gl}_n(k); k)$$

Derived character maps

Want: relate $H_\bullet(A, n)$ to more computable invariants.

Proposition (Berest-Khachatryan-Ramadoss)

For any algebra $A \in \mathbf{Alg}_k$ and any n there exists a **canonical** *derived character map*

$$\mathrm{Tr}_n(A)_\bullet \colon HC_\bullet(A) \to H_\bullet(A, n)^{\mathrm{GL}_n},$$

extending the original character map

$$\mathrm{Tr} \colon HC_0(A) = A/[A, A] \to A_n^{\mathrm{GL}_n}$$

Symmetric algebras

Goal: compute derived character maps for $A = \mathrm{Sym}(W)$.

For simplicity, **assume $\mathbf{n} = \mathbf{1}$** (i.e. only consider $H_\bullet(A, 1)$).

$\mathrm{Tr}(A)_\bullet$ factors through the *reduced* cyclic homology $\overline{HC}_\bullet(A)$.

For $A = \mathrm{Sym}(W)$,

$$\overline{HC}_i(A) \simeq \Omega^i(W)/d\Omega^{i-1}(W), \quad \Omega^i(W) \simeq \mathrm{Sym}(W) \otimes \Lambda^i(W)$$

Thus, we can think of $\mathrm{Tr}(A)_i$ as maps

$$\mathrm{Tr}(A)_i \colon \Omega^i(W) \to H_i(A, 1)$$

Example: $A = k[x, y]$

- $\mathrm{DRep}_1(k[x,y]) \simeq k[x,y,\lambda]$ with zero differential.
- The character $\mathrm{Tr}_0 \colon k[x,y] \to k[x,y,\lambda]$ is given by

$$\mathrm{Tr}_0(P(x,y)) = P(x,y)$$

  for any $P(x,y) \in k[x,y]$.

- The character $\mathrm{Tr}_1 \colon \Omega^1(A) \to k[x,y,\lambda]$ is given by

$$\mathrm{Tr}_1(P(x,y)dx + Q(x,y)dy) = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\lambda$$

$Tr(A)_1$ for $A = \mathrm{Sym}(W)$

For $A = \mathrm{Sym}(W) \simeq k[x_1, \ldots, x_m]$

$$\mathrm{DRep}_1(A) \simeq \mathrm{Sym}(W) \otimes \mathbf{Sym}\left(\underbrace{\Lambda^2(W)}_{\deg=1} \oplus \cdots \oplus \underbrace{\Lambda^m(W)}_{\deg=m-1}\right).$$

with **zero differential**, so $H_\bullet(A, 1) \simeq \mathrm{DRep}_1(A)$.

$$\lambda(v_1, v_2, \ldots, v_p) := v_1 \wedge v_2 \wedge \ldots \wedge v_p \in \underbrace{\Lambda^p(W)}_{\deg=p-1} \subset \mathrm{DRep}_1(A)$$

Proposition

For $\alpha = \sum P_i dx_i \in \Omega^1(A)$ the map $\mathrm{Tr}(A)_1$ is given by

$$\mathrm{Tr}(A)_1(\alpha) = \sum_{i<j} \left(\frac{\partial P_i}{\partial x_j} - \frac{\partial P_j}{\partial x_i} \right) \lambda(x_i, x_j) \in H_\bullet(A, 1)$$

Example: Tr_2 for $A = k[x, y, z]$

Take $\omega = Pdx \wedge dy + Qdy \wedge dz + Rdz \wedge dx \in \Omega^2(A)$.
Then $\mathrm{Tr}(A)_2(\omega)$ is given by

$$M\lambda(x, y, z) + M_y\lambda(x, y)\lambda(y, z) + M_z\lambda(y, z)\lambda(z, x) + M_x\lambda(z, x)\lambda(x, y),$$

where

$$M := P_z + Q_x + R_y$$

and for a polynomial F, F_q denotes $\frac{\partial F}{\partial q}$.
$\mathrm{Tr}(A)_2 = D \circ d_{dR}$, where

$$D = s^{-1} + \widetilde{D} \colon \Omega^3 \to H_\bullet(A, 1)$$

is a certain canonical differential operator on differential forms.

Abstract Chern–Simons forms

Let $\mathcal{A}$ be a **cohomologically** graded commutative DG algebra, $\mathfrak{g}$ a finite dimensional Lie algebra.
A $\mathfrak{g}$-valued **connection** is an element $\theta \in \mathcal{A}^1 \otimes \mathfrak{g}$.
Its **curvature** is $\Theta := d\theta + \frac{1}{2}[\theta, \theta]$, and Bianchi identity holds:

$$d\Theta = [\Theta, \theta]$$

If $P \in k[\mathfrak{g}]^{ad \mathfrak{g}}$, $\deg(P) = r$, for $\alpha \in \mathcal{A} \otimes \mathrm{Sym}^r(\mathfrak{g})$ define $P(\alpha) \in \mathcal{A}$ via

$$\mathcal{A} \otimes \mathrm{Sym}^r(\mathfrak{g}) \xrightarrow{\frac{1}{r!}\,\mathrm{id} \otimes \mathrm{ev}_P} \mathcal{A}$$

Then $P(\Theta^r) \in \mathcal{A}^{2r}$ is exact, and there exists $\mathrm{CS}_P(\theta) \in \mathcal{A}^{2r-1}$ such that $d\,\mathrm{CS}_P(\theta) = P(\Theta^r)$ with $\mathrm{CS}_P(\theta)$ is given explicitly by

$$\mathrm{CS}_P(\theta) = r! \int_0^1 P(\theta \wedge \Theta_t^{r-1}) dt$$

where $\Theta_t = t\Theta + \frac{1}{2}(t^2 - t)[\theta, \theta]$.

Derived character maps for polynomial algebras

Take $\mathcal{A} = \underline{\mathrm{hom}}(\Omega^\bullet(W), R_{ab})$, $\mathfrak{g} = k$, and $P_r = x^r \in k[\mathfrak{g}] \simeq k[x]$.

Theorem (Berest-Felder-P-Ramadoss-Willwacher)

There is a canonical k-valued connection θ in $\mathcal{A}$ such that the derived character map $\mathrm{Tr}(A)_\bullet \colon \Omega^\bullet(A) \to R_{ab} \simeq H_\bullet(A, 1)$ is given by

$$\mathrm{Tr}(A)_\bullet = \sum_{r=0}^{\infty} \mathrm{CS}_{P_r}(\theta) \circ d.$$

Here, $\theta(P(x_1, \ldots, x_m)dx_{i_1} \ldots dx_{i_p}) = P(0, \ldots, 0)\lambda(x_{i_1}, \ldots, x_{i_p})$

Remark: this allows to get explicit formulas for all derived character maps.

References

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