Isotropic Schur roots

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joint with Jerzy Weyman

Outline

- Describe the perpendicular category of an isotropic Schur root.
- Describe the cone of dimension vectors of the above.
- Describe the ring of semi-invariants of an isotropic Schur root.
- Construct all isotropic Schur roots.

Quivers, dimension vectors

- $k = \bar{k}$ is an algebraically closed field.
- $Q = (Q_0, Q_1)$ is an acyclic quiver with $Q_0 = \{1, 2, \ldots, n\}$.
- $\mathrm{rep}(Q)$ denotes the category of finite dimensional representations of Q over k.
- An element $\mathbf{d} \in (\mathbb{Z}_{\geq 0})^n$ is a dimension vector.
- Given $M \in \mathrm{rep}(Q)$, we denote by $\mathbf{d_M}$ its dimension vector.

Geometry of quivers

- For $\mathbf{d} = (d_1, \ldots, d_n)$ a dimension vector, denote by $\mathrm{rep}(Q, \mathbf{d})$ the set of representations M with $M(i) = k^{d_i}$.
- $\mathrm{rep}(Q, \mathbf{d})$ is an affine space.
- For such a $\mathbf{d}$, we set $\mathrm{GL}(\mathbf{d}) = \prod_{1 \leq i \leq n} \mathrm{GL}_{d_i}(k)$.
- The group $\mathrm{GL}(\mathbf{d})$ acts on $\mathrm{rep}(Q, \mathbf{d})$ and for $M \in \mathrm{rep}(Q, \mathbf{d})$ a representation, $\mathrm{GL}(\mathbf{d}) \cdot M$ is its isomorphism class in $\mathrm{rep}(Q, \mathbf{d})$.

Bilinear form and roots

- We denote by $\langle -, - \rangle$ the Euler-Ringel form of Q.
- For $M, N \in \mathrm{rep}(Q)$, we have

$$\langle \mathbf{d_M}, \mathbf{d_N} \rangle = \dim_k \mathrm{Hom}(M, N) - \dim_k \mathrm{Ext}^1(M, N).$$

Roots and Schur roots

- $\mathbf{d} \in (\mathbb{Z}_{\geq 0})^n$ is a (positive) root if $\mathbf{d} = \mathbf{d_M}$ for some indecomposable $M \in \mathrm{rep}(Q)$.
- Then $\langle \mathbf{d}, \mathbf{d} \rangle \leq 1$ and we call $\mathbf{d}$:

$$\begin{cases} \text{real}, & \text{if } \langle \mathbf{d}, \mathbf{d} \rangle = 1; \\ \text{isotropic}, & \text{if } \langle \mathbf{d}, \mathbf{d} \rangle = 0; \\ \text{imaginary}, & \text{if } \langle \mathbf{d}, \mathbf{d} \rangle < 0; \end{cases}$$

- A representation M is Schur if $\mathrm{End}(M) = k$.
- If M is a Schur representation, then $\mathbf{d_M}$ is a Schur root.
- We have real, isotropic and imaginary Schur roots.
- $\{$iso. classes of excep. repr.$\} \xleftrightarrow{1-1} \{$real Schur roots$\}$.

Perpendicular categories

- For $\mathbf{d}$ a dimension vector, we set $\mathcal{A}(\mathbf{d})$ the subcategory

$$\mathcal{A}(\mathbf{d}) = \{X \in \mathrm{rep}(Q) \mid \mathrm{Hom}(X,N) = 0 = \mathrm{Ext}^1(X,N)$$

$$\text{for some } \; N \in \mathrm{rep}(Q,\mathbf{d})\}.$$

- $\mathcal{A}(\mathbf{d})$ is an exact extension-closed abelian subcategory of $\mathrm{rep}(Q)$.
- If V is rigid (in particular, exceptional), then $\mathcal{A}(\mathbf{d_V}) = {}^{\perp}V$.

Proposition (-, Weyman)

For a dimension vector $\mathbf{d}$, $\mathcal{A}(\mathbf{d})$ is generated by an exceptional sequence $\Leftrightarrow$ $\mathbf{d}$ is the dimension vector of a rigid representation.

Perpendicular category of an isotropic Schur root

- Let $\boldsymbol{\delta}$ be an isotropic Schur root of Q (so $\langle \boldsymbol{\delta}, \boldsymbol{\delta} \rangle = 0$).

Proposition (-, Weyman)

There is an exceptional sequence $(M_{n-2}, \ldots, M_1)$ *in* $\mathrm{rep}(Q)$ *where all* M_i *are simples in* $\mathcal{A}(\boldsymbol{\delta})$.

- Complete this to a full exceptional sequence $(M_{n-2}, \ldots, M_1, V, W)$.

Perpendicular category of an isotropic Schur root

- We have $\{M_{n-2}, \dots, M_2, M_1\} = J \mathbin{\dot\cup} K \mathbin{\dot\cup} L$.
- For $I \subseteq \{M_{n-2}, \dots, M_2, M_1\}$, let $E(I)$ denote the corr. exceptional sequence.
- We consider the following:

$$(E(J \cup K), V, W).$$

- Now, the M_j and M_k are pairwise orthogonal. Thus, we get the following:

$$(E(J), E(K), V, W).$$

- Reflecting V, W yields:

$$(E(J), V', W', E(K)).$$

- Consider $\mathcal{R}(Q, \delta) := \mathrm{Thick}(E(J), V', W')$.

Perpendicular category of an isotropic Schur root

Theorem (-, Weyman)

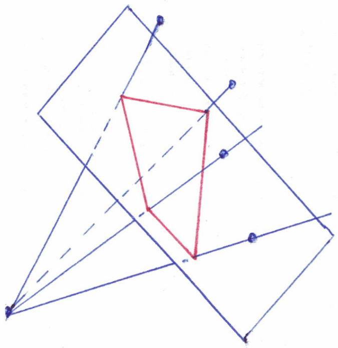
The category $\mathcal{R}(Q,\delta)$ is tame connected with isotropic Schur root $\bar{\boldsymbol{\delta}}$. It is uniquely determined by $(Q,\boldsymbol{\delta})$. The simple objects in $\mathcal{A}(\boldsymbol{\delta})$ are:

- *The M_i with $M_i \in K \cup L$,*
- *The quasi-simple objects of $\mathcal{R}(Q,\delta)$ (which includes the M_i with $M_i \in J$).*

- In particular, the dimension vectors of those simple objects are either $\bar{\boldsymbol{\delta}}$ or finitely many real Schur root.
- This gives the $-\langle -,\delta\rangle$-stable dimension vectors.

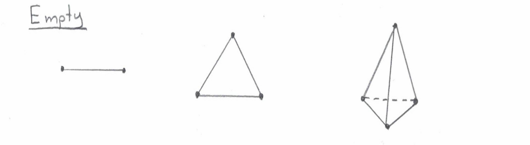
Cone of dimension vectors

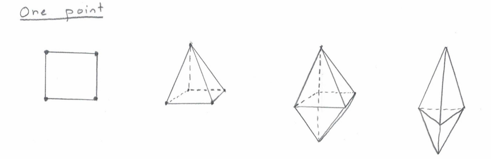
- Take the cone in $\mathbb{R}^n$ of all dimension vectors in $\mathcal{A}(\boldsymbol{\delta})$.
- This cone lives in dimension $n-1$ since it satisfies the equation $\langle -, \boldsymbol{\delta} \rangle = 0$.
- Take an affine slice of it. This becomes an $(n-2)$ dimensional polyhedron P.



Cone of dimension vectors

- Let V be the set of vertices of P and for $v \in V$, let P_v the convex hull of the points in $V \setminus \{v\}$.
- Only one dimension vector of simples is not real $\Rightarrow$ we have $|\bigcap_{v \in V} P_v| \leq 1$.
- Here are examples of such a cone in dimension $1, 2, 3$:





Cone of dimension vectors

Theorem (-, Weyman, *H. Thomas*)

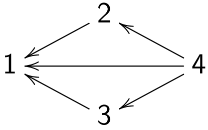
Assume that P is an $(n-2)$-dimensional convex hull of points V with $|\bigcap_{v \in V} P_v| \leq 1$. Then

$$\mathbb{R}^{n-2} = V_1 \oplus \cdots \oplus V_r$$

where V_i contains $\dim V_i + 1$ points in $V \cup \{0\}$ forming a $\dim V_i$-simplex containing the origin.

An example

Consider the quiver



- We take $\boldsymbol{\delta} = (3, 2, 3, 1)$.
- We get an exceptional sequence whose dimension vectors are $((8, 3, 3, 3), (0, 0, 1, 0), (0, 1, 0, 0), (3, 3, 3, 1))$.
- We have $\boldsymbol{\delta} = (3, 3, 3, 1) - (0, 1, 0, 0)$.
- $\boldsymbol{\delta}$ does not lie in the τ-orbit of $(1, 1, 0, 1)$ or $(1, 0, 1, 1)$.
- $\bar{\boldsymbol{\delta}} = (3, 2, 1, 1)$.
- Simple objects in $\mathcal{A}(\boldsymbol{\delta})$ are of dimension vectors $(0, 0, 1, 0), (8, 3, 3, 3)$ or $(3, 2, 1, 1)$.
- We have $\mathcal{R}(Q, \boldsymbol{\delta})$ of Kronecker type.

An example

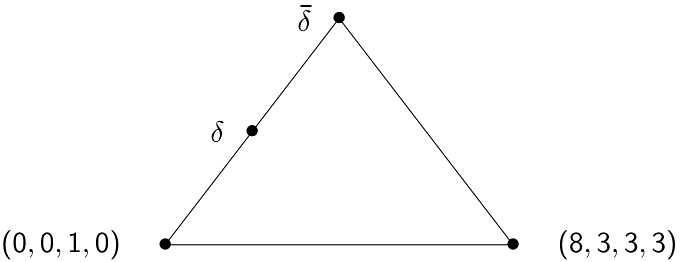


Figure : The cone of dimension vectors for $\delta = (3, 2, 3, 1)$

Semi-invariants

- Since Q is acyclic, the ring of invariants

$$k[\mathrm{rep}(Q,\mathbf{d})]^{\mathrm{GL}(\mathbf{d})}$$

  is trivial.
- Take $\mathrm{SL}(\mathbf{d}) = \prod_{1\le i\le n} \mathrm{SL}_{d_i}(k) \subset \mathrm{GL}(\mathbf{d})$.
- The ring $\mathrm{SI}(Q,\mathbf{d}) := k[\mathrm{rep}(Q,\mathbf{d})]^{\mathrm{SL}(\mathbf{d})}$ is the ring of semi-invariants of Q of dimension vector $\mathbf{d}$.
- We have $\mathrm{SI}(Q,\mathbf{d}) = \bigoplus_{\tau\in\Gamma} \mathrm{SI}(Q,\mathbf{d})_\tau$ where

$$\Gamma = \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}^n,\mathbb{Z}).$$

- This ring is always finitely generated.

Semi-invariants

- For $X \in \mathrm{rep}(Q)$, let $0 \to P_1 \xrightarrow{f_X} P_0 \to X \to 0$ be a projective resolution of X.
- For $M \in \mathrm{rep}(Q)$, the map

$$0 \to \mathrm{Hom}(X, M) \to \mathrm{Hom}(P_0, M) \xrightarrow{\mathrm{Hom}(f_X, M)} \mathrm{Hom}(P_1, M) \to \mathrm{Ext}^1(X, M) \to 0$$

is given by a square matrix $\Leftrightarrow \langle \mathbf{d_X}, \mathbf{d_M} \rangle = \mathbf{0}$.

- We set $C^X(M) := \det \mathrm{Hom}(f_X, M)$.

Proposition (Derksen-Weyman, Schofield-Van den Bergh)

The function $C^X(-)$ is a non-zero semi-invariant of weight $\langle \mathbf{d_X}, - \rangle$ in $\mathrm{SI}(Q, \mathbf{d})$ provided $\langle \mathbf{d_X}, \mathbf{d} \rangle = 0$. Moreover, these semi-invariants span $\mathrm{SI}(Q, \mathbf{d})$ over k.

Ring of semi-invariants of an isotropic Schur root

- The ring $\mathrm{SI}(Q, \mathbf{d})$ is generated by the $C^X(-)$ where X is simple in $\mathcal{A}(\mathbf{d})$.

Theorem (-, Weyman)

We have $\mathrm{SI}(Q, \boldsymbol{\delta}) \cong \mathrm{SI}(\mathcal{R}, \bar{\boldsymbol{\delta}})[\{C^{M_j}(-) \mid j \in K \cup L\}]$.

Corollary

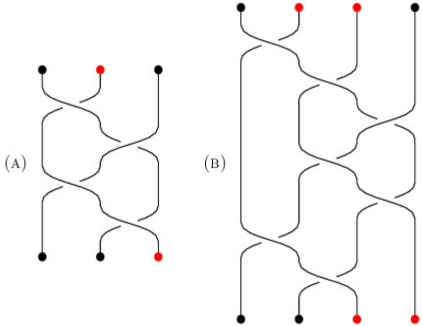
By a result of Skowroński - Weyman, $\mathrm{SI}(Q, \boldsymbol{\delta})$ *is a polynomial ring or a hypersurface.*

Exceptional sequences of isotropic types

- A full exceptional sequence $E = (X_1, \ldots, X_n)$ is of isotropic type if there are X_i, X_{i+1} such that $\text{Thick}(X_i, X_{i+1})$ is tame.
- Isotropic position is i and root type $\boldsymbol{\delta}_E$ is the unique iso. Schur root in $\text{Thick}(X_i, X_{i+1})$.
- The braid group B_n acts on full exceptional sequences.
- This induces an action of B_{n-1} on exceptional sequences of isotropic type.

An example

- Consider an exceptional sequence $E = (X, U, V, Y)$ of isotropic type with position 2.



- The exceptional sequence $E' = (X', Y', U', V')$ is of isotropic type with isotropic position 3.

Constructing isotropic Schur roots

Theorem (-, Weyman)

An orbit of exceptional sequences of isotropic type under B_{n-1} always contains a sequence E with δ_E an isotropic Schur root of an Euclidean full subquiver of Q.

Corollary

There are finitely many orbits under B_{n-1}.

- We can construct all isotropic Schur roots starting from the *easy ones*.

THANK YOU

Questions ?