Silting modules over commutative rings

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April 27, 2016

Silting modules

Let R be a ring. A right R-module T is called silting provided that there is a projective presentation

$$P_1 \xrightarrow{\sigma} P_0 \to T \to 0,$$

such that

$$\mathrm{Gen}(T) = \mathcal{D}_\sigma,$$

where

$$\mathrm{Gen}(T) = \{M \in \text{Mod-R} \mid \exists \varkappa : T^{(\varkappa)} \to M \to 0\},$$

$$\mathcal{D}_\sigma = \{M \in \text{Mod-R} \mid \mathrm{Hom}_R(\sigma, M) \text{ is surjective}\}.$$

Warning: The choice of projective presentation matters!
Introduced by Angeleri-Marks-Vitoria in 2014.

Examples

- Tilting modules (of pd ≤ 1, large, over arbitrary ring). A silting module is tilting precisely when σ can be chosen injective.
- Support τ-tilting modules over a finite dimensional algebra.
- Let R be von Neumann regular and commutative. Then $\{R/I \mid I \text{ an ideal}\}$ is a set of representatives of all silting R-modules up to equivalence.

Two silting modules T and T' are equivalent, if $\text{Gen}(T) = \text{Gen}(T')$. We call a class $\mathcal{C} \subseteq$ Mod-R silting if there is a silting module T such that $\mathcal{C} = \text{Gen}(T)$.

Motivation

1. Silting modules $\leftrightarrow$ 2-term silting complexes in the derived category (A-M-V).
2. Silting modules share many nice properties of tilting modules (Bongartz completion, silting classes are of finite type and definable).
3. Connections to various notions of localization (silting ring epimorphisms, smashing localizations over Prufer domain, abelian localizations,...).
4. Silting classes are definable torsion classes providing "non-monic" special preenvelopes (this is not a characterization).

Commutative setting

Goal: Classify silting classes/modules over arbitrary commutative ring.

Proposition (Angeleri, m.)

Let R be commutative and T a silting module. Then it is equivalent:

1. *T is projective,*
2. *T is equivalent to a f.p. silting module,*
3. $\operatorname{Ker}\operatorname{Hom}_R(T,-)$ *is closed under direct limits,*
4. $\operatorname{Gen}(T) = \operatorname{Gen}(Re)$ *for some idempotent $e \in R$.*

Grain of finiteness

Definition
A full subcategory $\mathcal{C}$ of Mod-R is definable if it is closed under pure submodules, products, and direct limits.

Theorem (Šťovíček-Marks, Angeleri-m., Bazzoni-Herbera)
Let $\sigma : P_1 \to P_0$ be a map between projectives. TFAE

1. *$\mathcal{D}_\sigma$ is a silting class,*
2. *$\mathcal{D}_\sigma$ is definable,*
3. *σ is of finite type.*

Finite type means of σ means: $\mathcal{D}_\sigma = \bigcup_{i \in I} \mathcal{D}_{\sigma_i}$, where σ_i's are maps between finitely generated projectives.

Dual setting

A left R-module C is called cosilting, if it admits an injective copresentation

$$0 \to C \to I_0 \xrightarrow{\lambda} I_1,$$

such that

$$\mathrm{Cogen}(C) = \mathcal{C}_\lambda,$$

where

$$\mathrm{Cogen}(C) = \{M \in \text{R-Mod} \mid \exists \varkappa : 0 \to M \to C^{\varkappa}\},$$

$$\mathcal{C}_\lambda = \{M \in \text{R-Mod} \mid \mathrm{Hom}_R(M, \lambda) \text{ is surjective}\}.$$

Dual definability

Let $(-)^+ = \mathrm{Hom}_{\mathbb{Z}}(-, \mathbb{Q}/\mathbb{Z})$ denote the elementary duality. Given a silting module T, the module T^+ is cosilting. Furthermore, the associated silting and cosilting classes are dual definable in the sense of M. Prest.

But, with respect to this duality, there are more cosilting than silting classes. To illustrate this:

- Cosilting classes = definable torsion-free classes (Breaz-Žemlička, Wei-Zhang),
- but not every definable torsion class is silting.

Thomason sets

Definable classes $\leftrightarrow$ closed subsets of Ziegler spectrum, $\mathrm{Zg}(R)$.
If R is commutative, there is a continous map $\mathrm{Zg}(R) \to \mathrm{Spec}(R)$.
But here, $\mathrm{Spec}(R)$ is endowed with not Zariski, but Hochster topology:
A subset X of $\mathrm{Spec}(R)$ is Thomason (Hochster) open, if X is a union of sets of form $V(I) = \{\mathfrak{p} \mid I \subseteq \mathfrak{p}\}$, where I is finitely generated ideal.

Theorem (Thomason, '97, Kock-Pitsch, '13)

Let R be a commutative ring. There is a (very nice) bijection between compactly generated localizing subcategories of $\mathcal{D}(R)$ and Thomason open subsets of $\mathrm{Spec}(R)$.

Main result

Theorem (Angeleri, m.)

Let R be a commutative ring. There is a bijection between:

1. *Silting classes $\mathcal{T}$ in* Mod-R,
2. *Thomason subsets X of* $\mathrm{Spec}(R)$.

The bijection works as follows:

$$\bigcup_{I \in \mathcal{I}} V(I) = X \mapsto \{M \in \text{Mod-R} \mid M = IM \ \forall I \in \mathcal{I}\}.$$

TL;DR: Silting classes over commutative rings = divisiblity classes by f.g. ideals.
Remarks:

- *Tilting classes correspond to those Thomason subsets avoiding* Ass(Flat-R).
- *Thomason sets $\leftrightarrow$ hereditary torsion pairs of finite type $\subseteq$ hereditary torsion pairs $\leftrightarrow$ abelian localizations!*

Noetherian setting

Theorem (Angeleri, m.)

Let R be a commutative noetherian ring. There is a bijection between:

1. *Silting classes $\mathcal{T}$ in* Mod-R,
2. *Upper sets X of* $\operatorname{Spec}(R)$.

The bijection works as follows:

$$X \mapsto \{M \in \text{Mod-R} \mid M = \mathfrak{p}M \ \forall \mathfrak{p} \in X\}.$$

Furthermore, (even for left noetherian non-commutative ring):

1. *all cosilting class in* R-Mod *are dual some silting class in* Mod-R,
2. *silting classes in* Mod-R $=$ *definable torsion classes.*

Flow of the proof

1. Silting module T.
2. Jump into the dual setting, cosilting module T^+.
3. Show that $\mathrm{Cogen}(T^+)$ is a torsion-free class of a hereditary torsion pair (here we need the commutativity!).
4. Gabriel theorem provides a corresponding Gabriel filter, which further corresponds to a Thomason set.
5. Show that Thomason set induces a silting class using Auslander-Bridger transpose and either:
   - finite type of silting classes (Šťovíček-Marks), or
   - explicit construction of the silting module (small object argument does not work directly here).

Thank you for your attention!

References

1. Lidia Angeleri Hügel, m., Silting modules over commutative rings, arXiv: 1602.04321.
2. m., One-tilting classes and modules over commutative rings, arXiv: 1507.02811.