The No Gap Conjecture: Proof by Pictures

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Maurice Auslander Distinguished Lectures
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Preliminaries

Joint With

- Kiyoshi Igusa. arXiv:1601.04054
- Thomas Brüstle, Kiyoshi Igusa and Gordana Todorov. arXiv:1503.07945

Notation/Conventions

- K denotes a (not necessarily algebraically closed) field,
- Λ a finite dimensional, basic, hereditary K-algebra
- with n indecomposable simples.

Mutation

- Quiver mutation introduced in the context of cluster algebras by Fomin-Zelevinsky.
- Categorified to an operation on collections of *exceptional objects* in the derived category $\mathcal{D}^b(\Lambda)$ by Buan-Marsh-Reiten-Reineke-Todorov.

Mutation

Definition

An object X in $\mathcal{D}^b(\Lambda)$ is *exceptional* if either:

1. $X = M$ is a module which is:
   - indecomposable and
   - rigid ($\mathrm{Ext}^1_\Lambda(M, M) = 0$)
2. $X = P_i[1]$ is a shift of an indecomposable projective module.

Remark

The exceptional objects form a fundamental domain for the cluster category $\mathcal{C}_\Lambda = \mathcal{D}^b(\Lambda)/\tau^- \circ [1]$.

Mutation

Mutation is defined through a *compatibility* relation on exceptional objects:

1. If M, N are modules, M and N are compatible whenever $\mathrm{Ext}^1_\Lambda(M, N) = 0$
2. $P_i[1]$ and M are compatible whenever $\mathrm{Hom}_\Lambda(P_i, M) = 0$
3. Each $P_i[1]$ and $P_j[1]$ are compatible.

Definition

A *cluster tilting object* is a maximal collection of compatible objects.

Mutation

Theorem (BMRRT)

1. *Every cluster tilting object* $T = T_1 \oplus \cdots \oplus T_n$ *has n direct summands.*
2. *For any* $1 \leq k \leq n$ *there is a unique* T'_k *not isomorphic to* T_k *so that* $T' = T/T_k \oplus T'_k$ *is a cluster tilting object.*

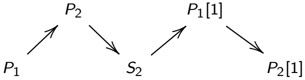
Definition

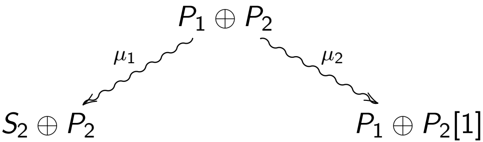
For a cluster tilting object T and $1 \leq k \leq n$, the *mutation* of T in the direction k is the cluster tilting object $\mu_k T \overset{\text{def}}{=} T'$.

Mutation

Example (Type $A_2 : 1 \leftarrow 2$)

AR Quiver:





Green Mutation

- Introduced by Keller for study of DT-invariants.
- Interpreted in context of representation theory by Ingalls-Thomas and Brüstle-Yang.
- Connections to weak order on Coxeter groups.

Definition (Brüstle-Dupont-Pérotin)

A mutation $\mu_k : T \mapsto T'$ is *green* (resp. *red*) if $\mathrm{Ext}^1_{\mathcal{D}^b(\Lambda)}(T'_k, T_k) \neq 0$ (resp. $\mathrm{Ext}^1_{\mathcal{D}^b(\Lambda)}(T_k, T'_k) \neq 0$).
Every mutation is either green or red.

Oriented Exchange Graph

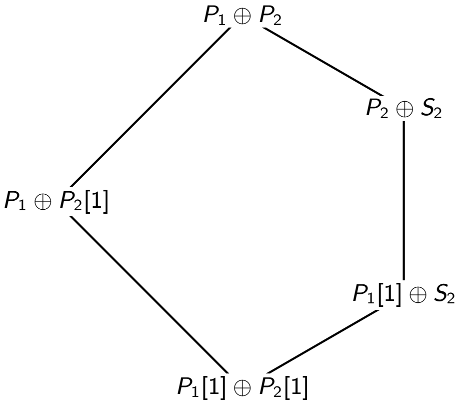
Green mutation makes set of cluster tilting objects $E(\Lambda)$ into a poset: $T \leq T'$ if T' obtained from T by a sequence of green mutations.

Properties

- unique minimal element $\Lambda[1]$ (every mutation green)
- unique maximal element Λ (every mutation red)
- No oriented cycles

Oriented Exchange Graph

Example (Type $A_2 : 1 \leftarrow 2$)



Maximal Green Sequences

Definition

A *maximal green sequence* is a (finite) sequence of green mutations starting with $\Lambda[1]$ and ending with Λ. Equivalently, a maximal (finite) chain in the poset $E(\Lambda)$.

Representation Theory Interpretation

There is a bijection $T \mapsto \mathrm{Fac}(T)$ between cluster tilting objects for Λ and *functorially finite* torsion classes. Cluster tilting objects satisfy $T \leq T'$ if and only if $\mathrm{Fac}(T) \subset \mathrm{Fac}(T')$.

The No Gap Conjecture

The No Gap Conjecture (Brüstle-Dupont-Pérotin)

The set of lengths of maximal green sequences for Λ forms an interval. That is, if Λ admits maximal green sequences of length ℓ and $\ell + k$, then there are maximal green sequences of lengths $\ell + i$ for all $0 \leq i \leq k$.

- Proven by Garver-McConville for:
  - Λ cluster tilted of type A_n
  - $\Lambda = KQ/I$ with Q oriented cycle
- Proven by Ryoichi Kase in type A_n and $\widetilde{A}_{1,n}$.

Remarks

- If $\Lambda = KQ$ where Q has oriented cycles, then it need not admit *any* maximal green sequences: e.g., quivers from once-punctured surfaces without boundary.
- Conjecture not true if K not algebraically closed. The (modulated) quiver B_2 has only two maximal green sequences: one of length 2 and the other of length 4.

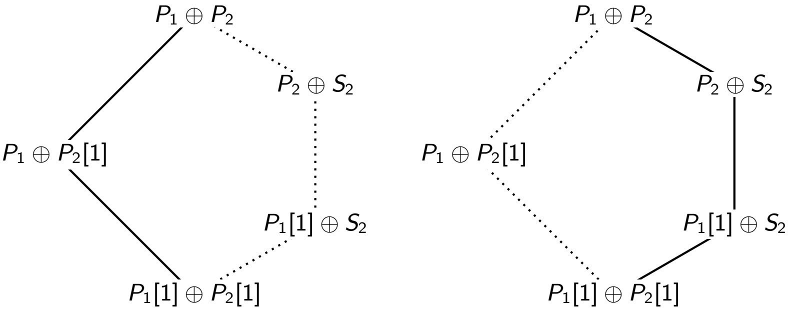
Polygonal Deformations

Definition

1. A *polygon* in $E(\Lambda)$ is a closed subgraph generated by two mutations μ_i, μ_j.
2. A *polygonal deformation* of a maximal green sequence is the operation of exchanging one side of a polygon in $E(\Lambda)$ for another.
3. Two maximal green sequences are *polygonally equivalent* if they differ by a sequence of polygonal deformations.

Polygonal Deformations

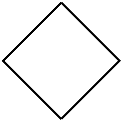
Example (Type $A_2 : 1 \leftarrow 2$)



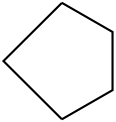
Polygonal Deformations

- If K algebraically closed, a (finite) polygon has either 4 or 5 edges. (If K arbitrary then can also have 6 or 8 sides.)
- If two maximal green sequences differ by a single polygonal deformation, their lengths differ by at most one.

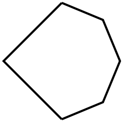
Polygons



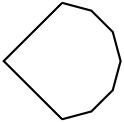
Type $A_1 \times A_1$



Type A_2



Type B_2



Type G_2

Polygonal Deformations

Theorem (H.-Igusa)

Let K be an arbitrary field. If Λ is tame, then any two maximal green sequences lie in the same polygonal deformation class. In particular, if K is algebraically closed the No Gap Conjecture is true for Λ.

Goal. Prove the Theorem using geometry of semi-invariant pictures.

Finding Maximal Green Sequences

Known for Λ tame there are only finitely many maximal green sequences (proven by BDP; different methods in BHIT). The Theorem implies an algorithm for finding *all* maximal green sequences for Λ:

1. Start with any maximal green sequence (e.g., shortest length).
2. Polygonally deform in all directions to get new maximal green sequences.
3. If return to a previous sequence, stop.

Finiteness implies this terminates. Theorem implies get all maximal green sequences.

Roots

Have the *Euler-Ringel* bilinear form

$$\langle\ ,\ \rangle : \mathbb{R}^n \otimes \mathbb{R}^n \to \mathbb{R}$$

given by $\langle \alpha, \beta \rangle = \alpha^t E \beta$ where

$$E_{ij} = \dim_K \mathrm{Hom}_\Lambda(S_i, S_j) - \dim_K \mathrm{Ext}_\Lambda(S_i, S_j)$$

Definition

A $\beta \in \mathbb{Z}^n$ is a *root* if there is an indecomposable β-dimensional representation of Λ. A root β is

1. *real* (resp. *null*) if $\langle \beta, \beta \rangle > 0$ (resp. $\langle \beta, \beta \rangle = 0$).
2. *Schur* if $\mathrm{End}(M) = K$ for some β-dimensional M.

The Cluster Fan

Fact
Real Schur roots in bijection with exceptional modules.

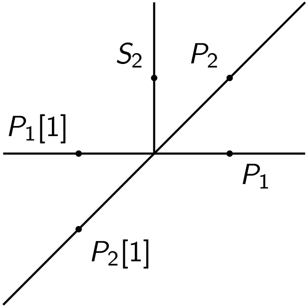
Definition
The *cluster fan* $F(\Lambda)$ is the simplicial fan generated by the rays $\mathbb{R}_{\geq 0}\beta$ in $\mathbb{R}^n$ where β either:

- a real Schur root
- negative a projective root.

A collection of rays span a cone in $F(\Lambda)$ whenever the corresponding exceptional objects are compatible.

The Cluster Fan

Example (Type $A_2 : 1 \leftarrow 2$)



The Cluster Fan

Definition
Let β be a real Schur root. The *semi-invariant domain*

$$D(\beta) = \{x \in \mathbb{R}^n : \langle x, \beta \rangle = 0 \text{ and } \langle x, \beta' \rangle \leq 0 \text{ for all } \beta' \subseteq \beta\}.$$

The walls (i.e., codim 1 cones) in $F(\Lambda)$ are the $D(\beta)$ for β real Schur root.

Theorem (Schofield, Derksen-Weyman, Igusa-Orr-Todorov-W)
The codimension 0 cones of $F(\Lambda)$ are in bijection with the cluster tilting objects for Λ. The cones corresponding to cluster tilting objects T and T' share a wall $D(\beta_k)$ if and only if $T' = \mu_k T$ with $\mathbf{dim}\, T_k = \beta_k$.

Question. The fan $F(\Lambda)$ gives geometric interpretation of cluster mutation. What about green/red mutation?

Semi-Invariant Pictures

Construction

1. Start with cluster fan $F(\Lambda)$ in $\mathbb{R}^n$.
2. Project real Schur roots β onto unit sphere S^{n-1}. Same for $\mathbf{dim}\Lambda[1]$.
3. Find hyperplane orthogonal to $\mathbf{dim}\Lambda$.
4. Stereographically project from $\mathbf{dim}\Lambda[1]$ onto hyperplane to get picture in $\mathbb{R}^{n-1}$.

The $D(\beta)$ become spherical segments in $\mathbb{R}^{n-1}$, with a *distinguished* normal orientation pointing towards $\mathbf{dim}\Lambda$.

Semi-Invariant Pictures

Example (Construction of Picture for $Q : 1 \leftarrow 2$)

1. Start with cluster fan $F(\Lambda)$ in $\mathbb{R}^n$.
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3. Find hyperplane orthogonal to **dim**Λ.
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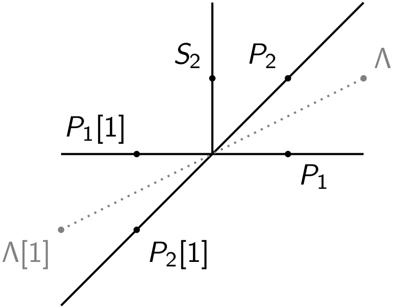

Semi-Invariant Pictures

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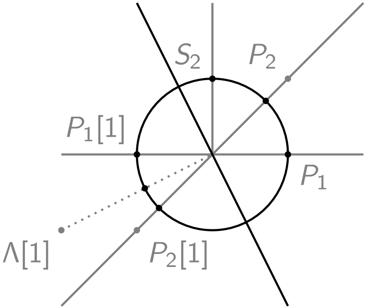

Semi-Invariant Pictures

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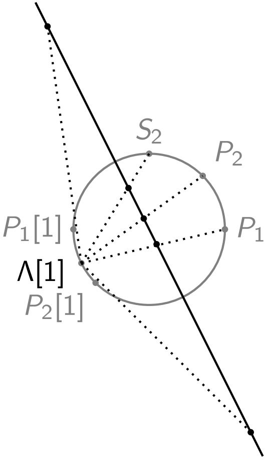

Semi-Invariant Pictures

Example (Construction of Picture for $Q : 1 \leftarrow 2$)

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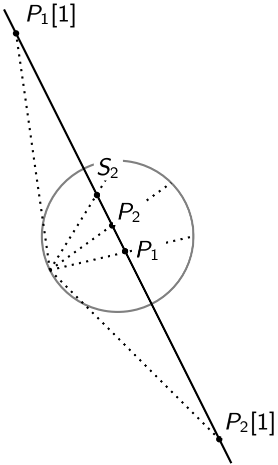
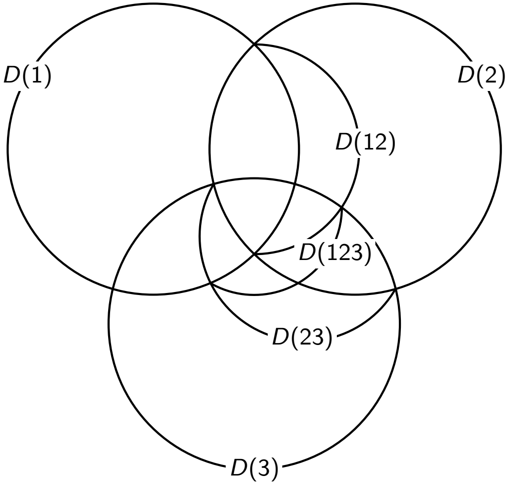


Semi-Invariant Pictures

Example (Picture for $Q : 1 \leftarrow 2 \leftarrow 3$)



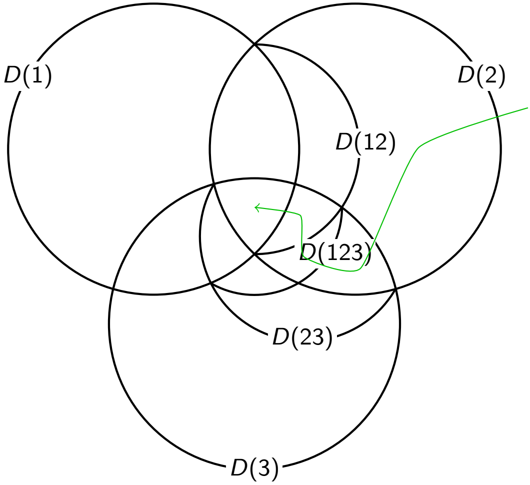
Semi-Invariant Pictures

Theorem (Igusa-Orr-Todorov-Weyman)

1. *Suppose* $\mu_k : T \mapsto T'$ *is a mutation, with corresponding cones* $C(T)$ *and* $C(T')$ *sharing the wall* $D(\beta_k)$*. The mutation* μ_k *is green if and only if* $C(T)$ *on outside of* $D(\beta_k)$ *and* $C(T')$ *on the inside.*
2. *Maximal green sequences are in bijection with (isotopy classes of) paths from* $C(\Lambda[1])$ *to* $C(\Lambda)$ *in* $L(\Lambda)$ *crossing walls* $D(\beta_k)$ *transversally from outside to inside.*

Semi-Invariant Pictures

Example (MGS for $Q : 1 \leftarrow 2 \leftarrow 3$)



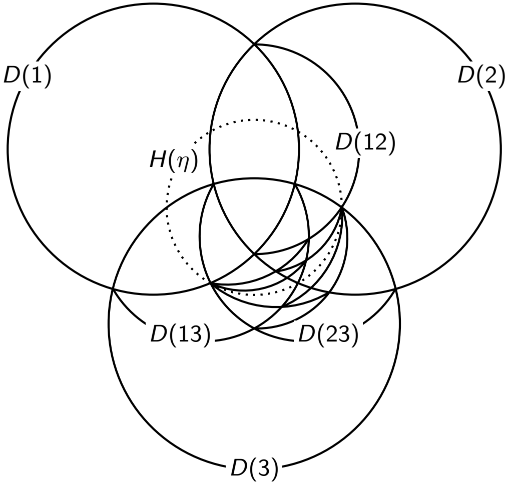
Semi-Regular Objects

For Λ tame, there is a unique minimal root η with $\langle \eta, \eta \rangle = 0$ called the *null root*.

- The set $H(\eta) = \{x \in \mathbb{R}^n : \langle x, \eta \rangle = 0\}$ is a hyperplane in $\mathbb{R}^n$. Gives an $(n-2)$-sphere in semi-invariant picture.
- Contains the domain $D(\eta) = \{x \in \mathbb{R}^n : \langle x, \alpha \rangle \leq 0$ for all preprojective $\alpha\}$.

Semi-Regular Objects

Example (Affine type A_3.)



Semi-Regular Objects

Definition

A cluster tilting object T is *semi-regular* if (the interior of) the cone $C(T)$ crosses $H(\eta) \setminus D(\eta)$.

Lemma (Brüstle-H.-Igusa-Todorov)

A cluster tilting object T is semi-regular if and only if it has

1. *a preprojective summand, and*
2. *a summand that is either preinjective or shifted projective.*

Proof of No Gap Conjecture (Sketch)

Step 1. Every maximal green sequence passes through some semi-regular object.

Step 2. Fix a semi-regular T. Any two maximal green sequence through T lie in the same deformation class.

Step 3. If T and T' are semi-regular, then there is a sequence of mutations

$$T = T^0, T^1, \ldots, T^k = T'$$

so that each T^i is semi-regular.

Thank You!