Convex Algebras

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General question

Given a rings surjection $\varphi\colon B \to A$ under what conditions is there a relationship between the homological properties of A and the homological properties of B.

By homological properties I mean projective resolutions, global dimension, and finitistic dimension. In general these properties do not behave well.

For example, let $A = K\mathcal{Q}/I$ for some admissible ideal I and assume that $\mathcal{Q}$ has at least one oriented cycle. Let J be the ideal generated by the arrows of $\mathcal{Q}$.

General question con't

Then, for some N, we have a surjection $\varphi\colon K\mathcal{Q}/J^N \to K\mathcal{Q}/I$.

Then finitisitic dimension of $K\mathcal{Q}/J^n$ is finite but unknown, in general, for A.

The global dimension of $K\mathcal{Q}/J^N$ is infinite but the global dimension of A,in general, can be any finite number or infinite.

Projective resolutions of simple $K\mathcal{Q}/J^N$-modules are reasonably well behaved but not much is known about projective resolutions of simple modules over an arbitrary ring A.

Convex subquivers

Joint work with Eduardo N. Marcos

$\mathcal{Q}$ is an arbitrary quiver.

$\mathcal{L}$ is a full subquiver of $\mathcal{Q}$. (All subquivers are assumed to be full).

For a while, we work only with quivers and the results will be independent of any relations.

We say a full subquiver $\mathcal{L}$ of $\mathcal{Q}$ is convex if every path from a vertex in $\mathcal{L}$ to a vertex in $\mathcal{L}$ lies in $\mathcal{L}$.

That is, if $p = v_1 \to v_2 \to \cdots \to v_n$ with $v_1, v_n \in \mathcal{L}_0$, then $v_i \in \mathcal{L}_0$ for $1 \leq i \leq n$.

The Convex Hull

Note that $\mathcal{Q}$ and the empty quiver are convex subquivers of $\mathcal{Q}$

The full subquiver with vertex set consisting of one vertex v is convex if and only if the only cycles through v are loops. The arrow set in this case is the set of loops at v.

If $\{\mathcal{L}_i\}$ is a collection of convex subquivers of $\mathcal{Q}$ then

$$\bigcap_i \mathcal{L}_i \text{ is a convex subquiver}$$

Thus, every subquiver of $\mathcal{Q}$ is contained in a unique smallest convex subquiver called the convex hull of $\mathcal{L}$.

The convex hull of a vertex $v \in \mathcal{Q}_0$ is the full subquiver of $\mathcal{Q}$ with vertex set consisting of the vertices that lie on an oriented cycle having v as one of its vertices.

Some subquiver constructions

Given a subquiver $\mathcal{L}$ of $\mathcal{Q}$, there are 3 important subquivers associated to it.

$$\mathcal{L}^+, \mathcal{L}^-, \mathcal{L}^o$$

The vertex set of $\mathcal{L}^+$ is the set of vertices v such that v is not in $\mathcal{L}$ and there is a path (in $\mathcal{Q}$) from a vertex in $\mathcal{L}$ to v.

The vertex set of $\mathcal{L}^-$ is the set of vertices v such that v is not in $\mathcal{L}$ and there is a path from v to a vertex in $\mathcal{L}$.

The vertex set of $\mathcal{L}^o$ is the set of vertices v such that v is not in $\mathcal{L}$ and there are no paths from or to v to or from a vertex in $\mathcal{L}$.

Properties

Some basic properties:

1. $\mathcal{Q} = \mathcal{L} \cup \mathcal{L}^+ \cup \mathcal{L}^- \cup \mathcal{L}^o$
2. If $\mathcal{L}$ is convex, then $\mathcal{L}^+, \mathcal{L}^-$, and $\mathcal{L}^o$ are convex.
3. $\mathcal{L} \cup \mathcal{L}^+$ and $\mathcal{L} \cup \mathcal{L}^-$ are convex.
4. $\mathcal{L}$ is convex if and only if $\mathcal{L}^+ \cap \mathcal{L}^-$ is empty.

Given a quiver $\mathcal{Q}$ and a vertex v in $\mathcal{Q}$, the path connected component of v is the full subquiver whose vertex set consisting of the vertices w such that both v and w lie on cycle. A path connected component is convex.
Note that the path component of a vertex v is the convex hull of the vertex v.

More properties

If either $\mathcal{L}^+$ or $\mathcal{L}^-$ is empty, then $\mathcal{L}$ is convex.
Proof If $\mathcal{L}^+$ is empty, then $\mathcal{L}^+ \cap \mathcal{L}^-$ is empty. Hence $\mathcal{L}$ is convex.

$(\mathcal{L} \cup \mathcal{L}^+)^+$ and $(\mathcal{L} \cup \mathcal{L}^-)^-$ are empty and hence $(\mathcal{L} \cup \mathcal{L}^+)$ and $(\mathcal{L} \cup \mathcal{L}^-)$ are convex.

Path connected components

We assume that the trivial path of length 0 consisting of a vertex v is considered to be a cyclic (the trivial cycle).
It is easy to see that if $\sim$ is the relation on the vertices of $\mathcal{Q}$ given by $v \sim w$ if v and w are vertices on some oriented cycle in $\mathcal{Q}$, then $\sim$ is a equivalence relation.
The equivalence classes of $\sim$ are the path connected components.

The trivial subquiver of $\mathcal{Q}$ at vertex v consists of one vertex, v and no arrows. The trivial subquiver at v is a path connected component if and only if v does not lie on an oriented cycle (of length ≥ 1).

Homological description of convexity

Proposition

Let $\mathcal{L}$ be a full subquiver of $\mathcal{Q}$ and $\Lambda = K\mathcal{Q}/J^2$, where J is the ideal in $K\mathcal{Q}$ generated by the arrows of $\mathcal{Q}$. The following statements are equivalent.

1. *$\mathcal{L}$ is not convex*
2. *There exist positive integers a and b and vertices u, v, w with $u, v \in \mathcal{L}_0$ and $w \notin \mathcal{L}_0$ such that both $\mathrm{Ext}^a_\Lambda(S_u, S_w)$ and $\mathrm{Ext}^b_\Lambda(S_w, S_v)$ are nonzero.*

Uses that since $\Lambda = K\mathcal{Q}/J^2$ is a Koszul algebra $\mathrm{Ext}^n_\Lambda(S_u, S_v)$ corresponds to a path of length n in $\mathcal{Q}$ from u to v.

We give another algebraic description of convexity later.

Algebras

Let $\Lambda = K\mathcal{Q}/I$ be a K-algebra.
K is an arbitrary field and I is an ideal contained in ideal generated by paths of length 2 in $K\mathcal{Q}$.

Let $\mathcal{L}$ be a full subquiver of $\mathcal{Q}$.

Let e be idempotent in $K\mathcal{Q}$ or Λ corresponding to the sum of the vertices in $\mathcal{L}$.
Let e' be idempotent in $K\mathcal{Q}$ or Λ corresponding to the sum of the vertices not in $\mathcal{L}$.

The algebra associated to $\mathcal{L}$ and Λ is $\Gamma = \Lambda/(\Lambda e' \Lambda)$.

The algebra assoc to $\mathcal{L}$ and Λ

An equally fine choice could have been $e\Lambda e$.

We have surjections:

$$\Lambda \to e\Lambda e \text{ given by } \lambda \to e\lambda e$$

and

$$\Lambda \to \Lambda/(\Lambda e'\Lambda), \text{ the canonical surjection.}$$

The first map is not a ring homomorphism in general.

$\Gamma = \Lambda/(\Lambda e'\Lambda)$ and $e\Lambda e$ are, in general not isomorphic as algebras.

Example

Convexity

Lemma

Suppose that $\mathcal{L}$ is a convex subquiver of $\mathcal{Q}$. Then if $\lambda, \gamma \in \Lambda$, $e\lambda e\gamma e = e\lambda\gamma e$. In particular, $e\lambda e'\gamma e = 0$.

Note that if $\mathcal{L}$ is convex, then the map $\Lambda \to e\Lambda e$, given by λ is sent to $e\lambda e$, is a ring homorphism.

There is a splitting of this homomorphism, namely the inclusion $e\Lambda e \to \Lambda$. This is a splitting as rings without identity .

Proposition

If $\mathcal{L}$ is convex then $\Gamma = \Lambda/(\Lambda e'\Lambda)$ is isomorphic to $e\Lambda e$, sending $\bar{\lambda}$ to $e\lambda e$, where $\lambda \in \Lambda$ and $\bar{\lambda}$ denotes the image of λ in $\Lambda/(\Lambda e'\Lambda)$.

A few references

Auslander, Platzeck, Todorov Idempotent ideals TAMS 1992

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Psaroudakis, Chrysostomos; Skartsaeterhagen, Oeystein Ingmar; Solberg, Oeyvind. Gorenstein categories, singular equivalences and finite generation of cohomology rings in recollements, TAMS

Diracca, Luca; Koenig, Steffen. Cohomological reduction by split pairs. J. Pure Appl. Algebra 212 (2008), no. 3, 471-485.

Another description of convexity

Let $I = (0)$. Thus $\Lambda = K\mathcal{Q}$. Now suppose that $\mathcal{L}$ is a subquiver of $\mathcal{Q}$ and let Γ be the algebra associated to $\mathcal{L}$ and Λ Then the following statements are equivalent.

1. $\mathcal{L}$ is a convex

2. $e\Lambda e$ is isomorphic to Γ.

Case of $\mathcal{L}^+$ empty

For the remainder of this talk $\Lambda = K\mathcal{Q}/I$, $\mathcal{L}$ is a full subquiver of $\mathcal{Q}$, and $\Gamma = \Lambda/(\Lambda e'\Lambda)$ is the algebra associated to $\mathcal{L}$ and Λ.

Proposition

Suppose that $\mathcal{L}^+$ is empty. If P is a projective Γ-module, then P is a projective Λ-module. Furthermore, if $M, N \in \mathbf{Mod}(\Gamma)$, then $Hom_\Gamma(M, N) = Hom_\Lambda(M, N)$

Relation to idempotent ideals -APT

Assuming $\mathcal{L}^+$ is empty, one can show that $\Lambda e'\Lambda = e'\Lambda$ and hence $\Lambda e'\Lambda$ is a strong idempotent ideal. Parts (2) and (3) below have been observed by Auslander, Platzeck, and Todorov (under the assumption that Λ is an artin algebra).

In APT, the duality between left and right modules is used. In the previous result and the following result, we do not assume that Λ is finite dimensional.

$\mathcal{L}^+$ empty con't

Theorem

Suppose that $\mathcal{L}^+$ is empty. The following statements hold.

1. *If $(*) : \cdots \to P^2 \to P^1 \to P^0 \to M \to 0$ is a projective Γ-resolution of the Γ-module M, then applying the forgetful functor$(*)$ is a projective Λ-resolution of M. If $(*)$ is minimal over Γ then $(*)$ is minimal over Λ.*

2. *If M and N are Γ-modules, then the Ext-algebra $\mathrm{Ext}^*_\Gamma(M,N)$ is graded isomorphic $\mathrm{Ext}^*_\Lambda(M,N)$. That is, $\mathbf{Mod}(\Gamma) \to \mathbf{Mod}(\Lambda)$ is a homological embedding.*

3. *gl.dim$(\Lambda) \geq$ gl.dim(Γ).*

4. *If Λ satisfies the finitistic dimension conjecture, so does Γ.*

There are similar results if $\mathcal{L}^-$ is empty.

Convexity Result

Theorem

Let K be a field, $\mathcal{Q}$ a finite quiver, and $\Lambda = K\mathcal{Q}/I$, where I is an ideal in $K\mathcal{Q}$ contained in ideal generated by paths of length 2 in $\mathcal{Q}$. Suppose that $\mathcal{L}$ is a convex subquiver of $\mathcal{Q}$ and let Γ be the algebra associated to Λ and $\mathcal{L}$. Then

1. $\mathrm{Ext}^*_\Lambda(M, N)$ *is graded isomorphic to* $\mathrm{Ext}^*_\Gamma(M, N)$*, for all* Γ*-modules* M *and* N*.*
2. $gl.dim(\Lambda) \geq gl.dim(\Gamma)$.
3. *The finitistic dimension of* $\Lambda \geq$ *the finitistic dimension of* Γ*.*

Hochschild cohomology

$\Lambda = K\mathcal{Q}/I$, where I is an admissible ideal in $K\mathcal{Q}$.

It is well-known that $\Lambda^e = \Lambda^{op} \otimes_K \Lambda = K\mathcal{Q}^*/I^*$ where $\mathcal{Q}^*$ is the quiver with vertex set $\mathcal{Q}^{\mathrm{op}} \times \mathcal{Q}$ where $\mathcal{Q}^{\mathrm{op}} = \{v^{\mathrm{op}} \mid v \in \mathcal{Q}_0\}$ and arrow set
$\{(a^{\mathrm{op}}, v) \mid a \in \mathcal{Q}_1, v \in \mathcal{Q}_0\} \cup \{(v^{\mathrm{op}}, a) \mid v^{\mathrm{op}} \in \mathcal{Q}_0^{\mathrm{op}}, a \in \mathcal{Q}_1\}$,
where $a^{\mathrm{op}} \colon v^{\mathrm{op}} \to w^{\mathrm{op}}$ if $a \colon w \to v$.

The ideal I^* is generated by the elements of the form $r^{\mathrm{op}} \otimes 1$ and $1 \otimes r'$, where $r^{\mathrm{op}} \in I^{\mathrm{op}}$ and $r' \in I$ together with commutativity relations coming from the tensor product $\Lambda^{\mathrm{op}} \otimes_K \Lambda$.

Convexity in this setting

Lemma

Suppose that $\mathcal{L}$ is a convex subquiver of $\mathcal{Q}$. Then $\mathcal{L}^{op} \times \mathcal{L}$ is a convex subquiver of $\mathcal{Q}^*$.

Let Γ be the algebra associated to Λ and $\mathcal{L}$. The algebra associated to Λ^e and $\mathcal{L}^{op} \times \mathcal{L}$ is isomorphic to Γ^e.

Theorem

Let K be a field, $\mathcal{Q}$ a finite quiver, and $\Lambda = K\mathcal{Q}/I$, where I is an admissible ideal in $K\mathcal{Q}$. Suppose that $\mathcal{L}$ is a convex subquiver of $\mathcal{Q}$ and let Γ be the algebra associated to Λ and $\mathcal{L}$. Then $\mathrm{Ext}^*_{\Lambda^e}(\Gamma, N)$ is graded isomorphic to $HH^*(\Gamma, N)$, for all Γ-bimodules N. In particular, $\mathrm{Ext}^*_{\Lambda^e}(\Gamma, \Gamma)$ is graded isomorphic to $HH^*(\Gamma)$.

The homological heart of an algebra

Let

$$X = \{v \in \mathcal{Q}_0 \mid v \text{ is a vertex in a nontrivial cycle in } \mathcal{Q}\}$$

and let

$$Y = X \cup \{y \in \mathcal{Q}_0 \mid y \text{ is a vertex in a path with origin}$$
$$\text{and terminus vertices in } X\}.$$

Let $\mathcal{H}(\mathcal{Q})$, or simply $\mathcal{H}$ when no confusion could arise, be the subquiver of $\mathcal{Q}$ with vertex set Y. We call $\mathcal{H}$ the homological heart of $\mathcal{Q}$.

Note that $\mathcal{H}$ depends only on $\mathcal{Q}$.

Basic properties

Proposition

Let $\mathcal{H}$ be the homological heart of $\mathcal{Q}$. Then the following statements hold.

1. *The subquiver $\mathcal{H}$ is the empty quiver if and only if $\mathcal{Q}$ contains no nontrivial cycles; that is, $\mathcal{Q}$ is triangular.*
2. *The subquiver $\mathcal{H}$ is the convex hull of X.*
3. *The quiver $\mathcal{H}$ is the smallest convex subquiver of $\mathcal{Q}$ that contains all the nontrivial path connected components of $\mathcal{Q}$.*
4. *The homological heart of $\mathcal{Q}$ is an invariant of $\mathcal{Q}$.*
5. *The subquiver $\mathcal{H}^+ \cup \mathcal{H}^- \cup \mathcal{H}^0$ contains no oriented cycles.*

Useful results

We set t to be the longest path in $\mathcal{Q}$ with support in $\mathcal{H}^- \cup \mathcal{H}^0 \cup \mathcal{H}^+$.

Lemma

Let M be a Λ-module. Then, for $\ell \geq t$

1. *the ℓ-th syzygy of a Λ-module has support in $\mathcal{H} \cup \mathcal{H}^+$ and*
2. *the ℓ-th cosyzygy of Λ-module has support in $\mathcal{H} \cup \mathcal{H}^-$.*

Let C be a Λ-module. Define C^+ to be the largest submodule of C having support contained in $\mathcal{H}^+$ and C_- be the smallest submodule of C such that C/C_- has support contained $\mathcal{H}^-$.

Main result about homological hearts

First a proposition.

Proposition

Let A be a Λ-module whose support is contained in $\mathcal{H} \cup \mathcal{H}^+$. If $\cdots \to P^2 \to P^1 \to P^0 \to A \to 0$ is a minimal projective Λ-resolution of A, then

$$\cdots \to P^2/(P^2)^+ \to P^1/(P^1)^+ \to P^0/(P^0)^+ \to A/A^+ \to 0$$

is a minimal projective Γ-resolution of A/A^+.

Theorem

Let K be a field, $\mathcal{Q}$ a finite quiver, and $\Lambda = K\mathcal{Q}/I$, where I is an admissible ideal I in $K\mathcal{Q}$. Let $\mathcal{H}$ be the homological heart of $\mathcal{Q}$ and Γ be the algebra associated to Λ and $\mathcal{H}$.

Theorem (Con't)

Let t be length of the longest path in $\mathcal{Q}$ having support contained in $\mathcal{H}^+ \cup \mathcal{H}^- \cup \mathcal{H}^0$.

Then

1. *gl.dim(Λ) is finite if and only if gl.dim(Γ) is finite.*
2. *The finitistic dimension of Λ is finite if and only if the finitistic dimension of Γ is finite.*
3. *If M and N are Λ-modules and $\ell \geq 2t+1$, then*

   $\mathrm{Ext}^{\ell}_{\Lambda}(M, N)$ *is naturally isomorphic to* $\mathrm{Ext}^{\ell-2t}_{\Gamma}(A_M, B_N)$,

   where $A_M = \Omega^t(M)/(\Omega^t(M))^+$ and $B_N = \Omega^{-t}(N)_-$.