A McKay correspondence for reflections groups

joint work with Ragnar-Olaf Buchweitz and Colin Ingalls

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Kleinian singularities

Focus on $n = 2$, and $k = \mathbb{C}$. Then

Theorem (F. Klein, 1884)

Let $\Gamma \subseteq SL_2(\mathbb{C})$ *be a finite group. Then the quotient singularity* $X = \mathbb{C}^2/\Gamma = \text{Spec}(S^\Gamma)$*, i.e., the orbit space of* Γ *acting on* $\mathbb{C}^2$*, is of the form*

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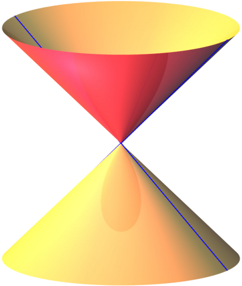
$$X = \mathrm{Spec}(\mathbb{C}[x, y, z]/(f)),$$

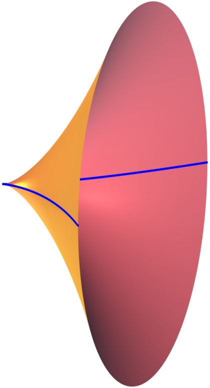
where f is of type

- A_n: $z^2 + y^2 + x^{n+1}$,
- D_n: $z^2 + x(y^2 + x^{n-2})$ for $n \geq 4$,
- E_6: $z^2 + x^3 + y^4$,
- E_7: $z^2 + x(x^2 + y^3)$,
- E_8: $z^2 + x^3 + y^5$.

A_1 and A_2 – the cone and the cusp

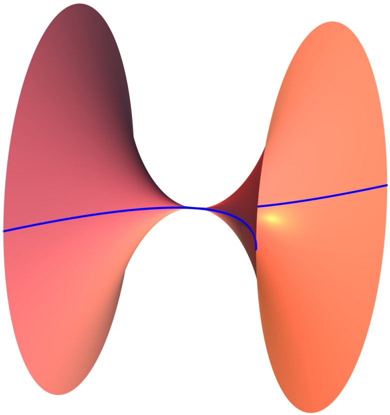


$$x^2 + y^2 - z^2 = 0$$



$$z^2 + y^2 - x^3 = 0$$

A_3 and A_4



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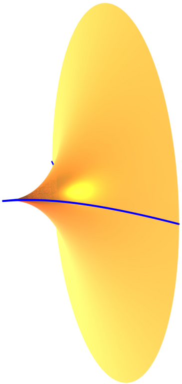

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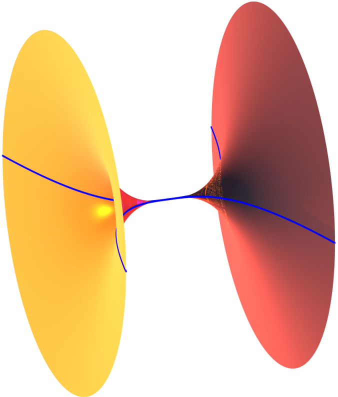
A_5 and A_6



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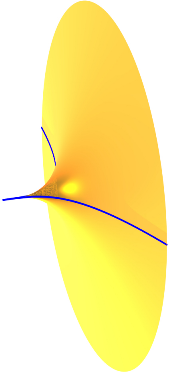
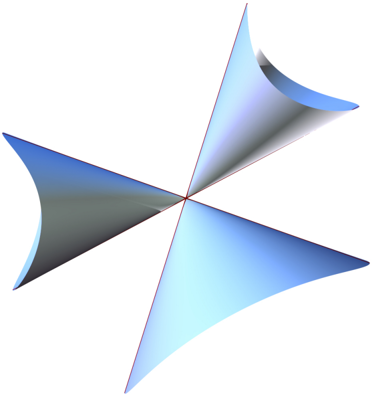
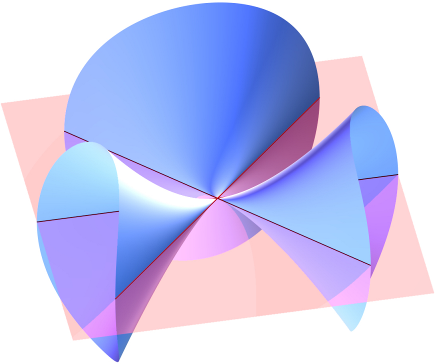


$z^2 + y^2 - x^7 = 0$

$D4 : z^2 + x(y^2 - x^2) = 0$





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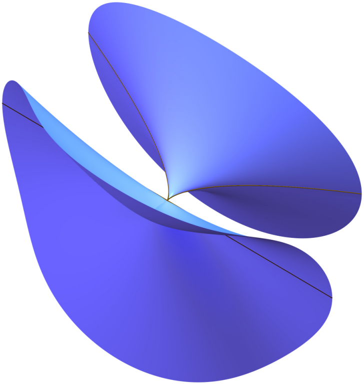

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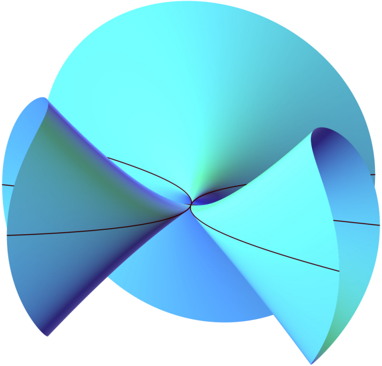

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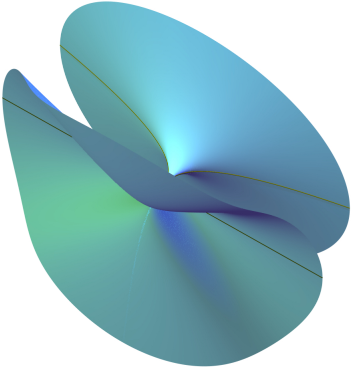
D_7 and D_8



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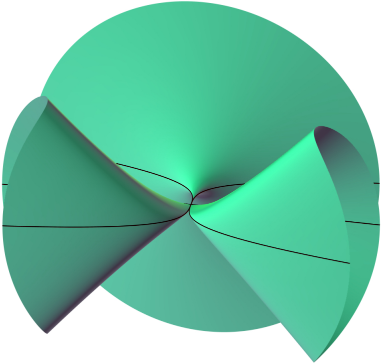

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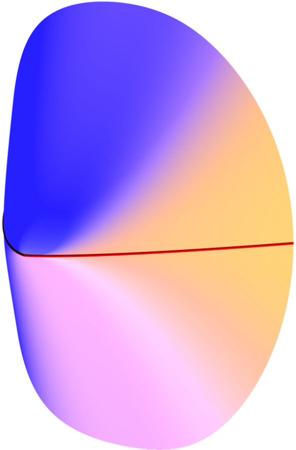
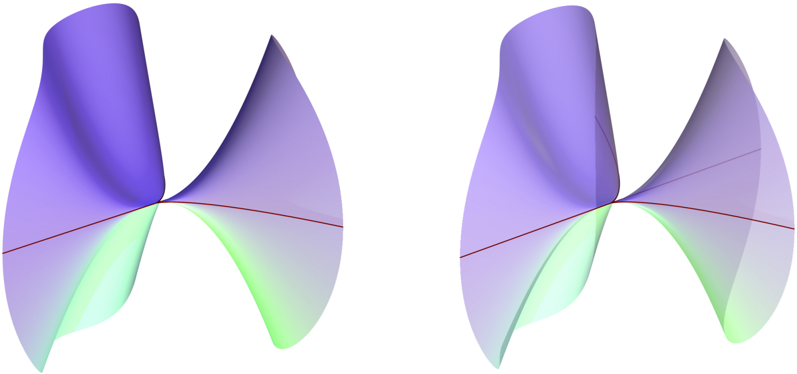
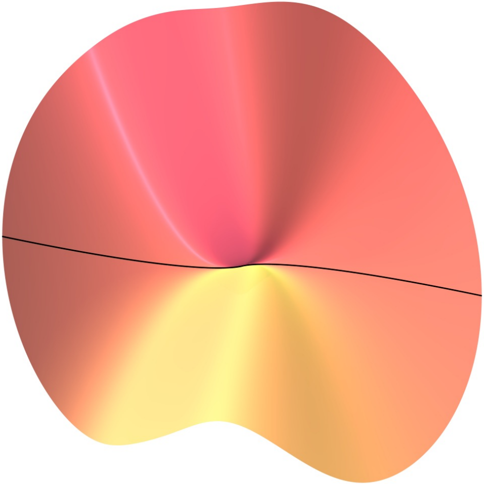
$E_6: \quad z^2 + x^3 + y^4 = 0$



$E_7: \quad z^2 + x(x^2 + y^3) = 0$



$E_8 : \quad z^2 + x^3 + y^5 = 0$



Dual resolution graphs

Let X be a normal surface singularity and let $\pi : \widetilde{X} \longrightarrow X$ be its minimal resolution, with exceptional curves $\bigcup_i E_i$.

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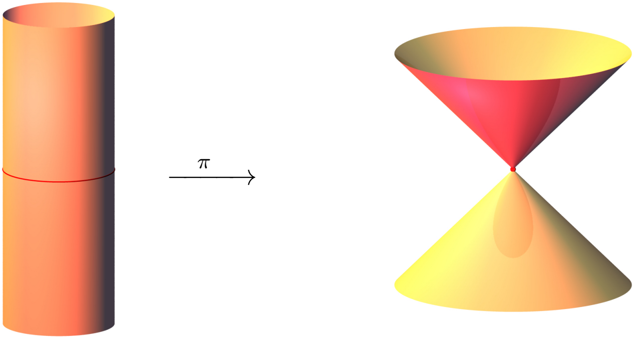
Theorem (Du Val)

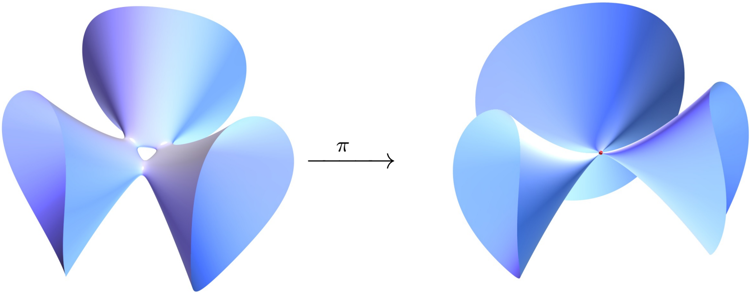
The dual resolution resolution graphs of the Kleinian singularities are Coxeter–Dynkin diagrams of type ADE.

Example: $x^2 + y^2 = z^2$



Dual resolution graph of type A_1:

Example: $z^2 + x(y^2 - x^2) = 0$



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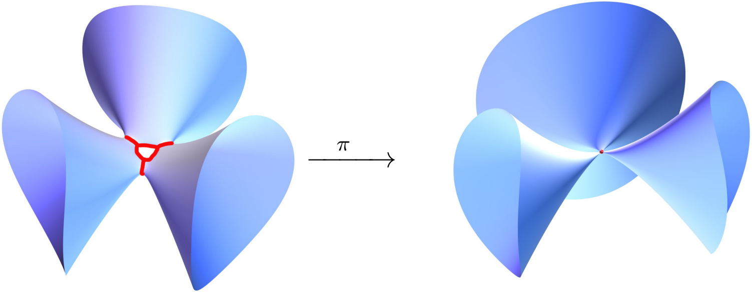
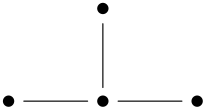


Dual resolution graph of type D_4:



McKay correspondence

Let $\Gamma \subseteq SL_2(\mathbb{C})$ be a finite group with irreducible representations $\rho_0, \ldots \rho_m$:
ρ_0 = trivial representation,
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Observation (J. McKay, 1979): These graphs are extended Coxeter Dynkin diagrams of type ADE (with arrows in both directions).

Example: D_4

The group Γ is generated by

$$\pm\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \pm\begin{pmatrix}i & 0\\ 0 & -i\end{pmatrix}, \pm\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}, \pm\begin{pmatrix}0 & i\\ i & 0\end{pmatrix}.$$

Five irreps ρ_i, four one-dimensional and one two-dimensional $\rho_1 = c$.

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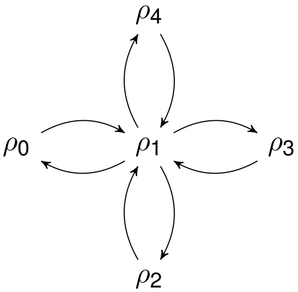
The McKay graph:



McKay correspondence

Thus for $n = 2$ and $\Gamma \in SL_2(\mathbb{C})$:
Have 1-1 correspondence between

- exceptional curves E_i on the minimal resolution of $\mathbb{C}^2/\Gamma$.
- irreducible representations of Γ (mod the trivial representation).
- indecomposable projective $\Gamma * S = \text{End}_R S$-modules (modulo the trivial module).
- indecomposable **CM**-modules over R (modulo R itself). [This follows from *Herzog's theorem*, which says that $\text{add}_R(S) = \mathbf{CM}(R)$.]

Theorem (Buchweitz–F–Ingalls)

If $G \subseteq GL_2(\mathbb{C})$ is a reflection group, let $z = \prod_{s \in reflections(G)} l_s$ be the hyperplane arrangement and set $\Delta = z^2$.
Let further $A = G * S$, $e = \frac{1}{|G|} \sum_{g \in G} g$, $\bar{A} = A/AeA$ and $T = S^G$. Then

$$\bar{A} \cong \mathrm{End}_{T/\Delta}(S/z)$$

is a NCR of T/Δ, that is, $\mathrm{gldim}\, \bar{A} = 2$ and S/z is in $\mathbf{CM}(T/\Delta)$.

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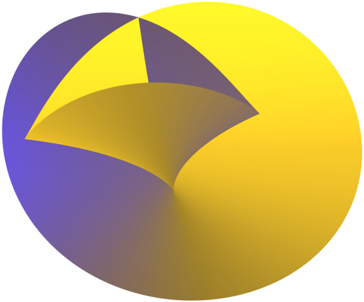
In particular:

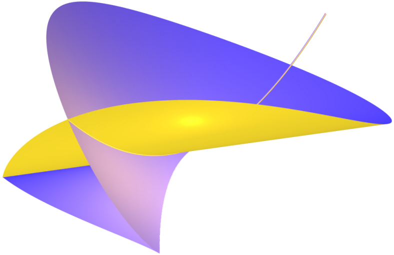
$$\mathrm{add}_{T/\Delta}(S/z) = \mathbf{CM}(T/\Delta),$$

*i.e., S/z is a **CM**-representation generator.*

The swallowtail: Δ of S_4

$$16x^4z - 4x^3y^2 - 128x^2z^2 + 144xy^2z - 27y^4 + 256z^3 = 0$$





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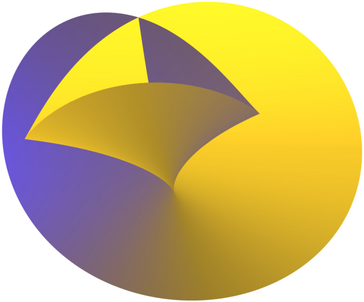

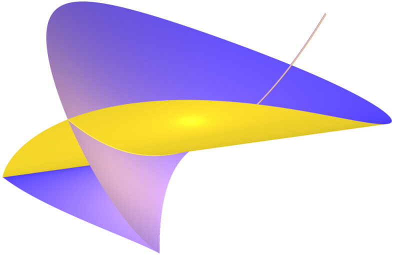

Here $S/z \cong T/\Delta \oplus \widetilde{T/\Delta} \oplus \text{syz}(\widetilde{T/\Delta}) \oplus M^2_{2,0}$.

Questions

- What are the R-direct summands of S/z?
- Can one describe the R-direct summands of S/z for some specific groups, e.g., S_n?
- What about the geometry?