Auslander-Reiten sequences and intuition

Gabriella D’Este

Thank you very much to
- all the Organizers
- all the Participants

Special thanks to Gordana Todorov

I am happy to participate also this year to this annual event, which was created to celebrate the **mathematical legacy** of **Maurice Auslander**, open to participants of any generation and country.

Why this title ?

1st reason: This = **Auslander Conference**

2nd reason: **AR - quivers** and **sequences** are **beautiful**

(for mathematicians & non mathematicians)

3rd reason: I used **AR - quivers** many times & from the beginning.

Where I met Auslander - Reiten sequences :

in **BIELEFELD** in the notes written by **Dieter Happel** (= Distinguished Speaker of the Auslander Lectures in 2009) of the lectures given by **Claus Michael Ringel** (= **.....** 2004)

More precise reference :

Vorlesungausarbeitung
Darstellungstheorie
endlich - dimensionaler Algebren,
Sommersemester 1979 ,
Universitaet Bielefeld

beautiful =

word taken from a previous home page of **NEU** on **Maurice Auslander Distinguished Lectures & International Conference.**

The first words are:

"Maurice Auslander was a mathematician who created influential and **beautiful** mathematics".

For the relationship between

mathematics and beauty, I recall the paper by **KARIN BAUR** and **KLEMENS FELLNER** , entitled "Mathematics and Arts. Towards a balance between artistic intuition and mathematical complexity" .

Personal experience with quite different people (children, …)

AR - quivers & sequences have **so many aspects** that the best definitions and theorems cannot describe them completely.

Fact: many mathematical objects have more than one shape, as observed in the www page of **CRC 701**:

CRC 701: Spectral Structures and Topological Methods in Mathematics

The CRC 701 pursues the vision of reinforcing and building bridges between various branches of theoretical and applied mathematics. The guiding principles in this undertaking are the investigation of **spectral structures** and the development and application of topological methods throughout mathematics and related sciences.

Spectral structures are omnipresent in mathematics and many of its application areas.

Perhaps, this is the reason why the word **"intuition"** shows up at the end of **GABRIEL** 's paper mentioned in the abstract of my talk.

GABRIEL , " Auslander - Reiten sequences and repres. - finite algebras", LMN 831]

The **name** is a dedication of Ringel to the authors of "almost split sequences". He introduced Auslander - Reiten quivers in his Brandeis lectures (1975) and determined their structure for tame and wild quivers [31] . Since then, various specialists like Bautista, Brenner, Butler, Riedtmann . . . . have hoarded a few hundred examples in their dossiers, thus getting an intuition which no theoretical argument can replace.

Many equivalent equivalent conditions
in Propositions 1.14 and 2.2 in
Auslander - Reiten - Smalo’s book,
pages 144 - 148 .

(ii) *f is a minimal right almost split morphism.* □

Propositions 1.12 and 1.13 suggest the following definition.

An exact sequence $0 \to A \xrightarrow{g} B \xrightarrow{f} C \to 0$ is called an **almost split sequence** if g is left almost split and f is right almost split. It is clear that an exact sequence $0 \to A \xrightarrow{g} B \xrightarrow{f} C \to 0$ is almost split if and only if $0 \to D(C) \xrightarrow{D(f)} D(B) \xrightarrow{D(g)} D(A) \to 0$ is almost split. Summarizing some of our previous results we have the following.

Proposition 1.14 *The following are equivalent for an exact sequence $0 \to A \xrightarrow{g} B \xrightarrow{f} C \to 0$.*

7 conditions

(a) *The sequence is an almost split sequence.*
(b) *The morphism f is minimal right almost split.*
(c) *The morphism g is minimal left almost split.*
(d) *The module A is indecomposable and f is right almost split.*
(e) *The module C is indecomposable and g is left almost split.*
(f) *The module C is isomorphic to $\operatorname{Tr} DA$ and g is left almost split.*
(g) *The module A is isomorphic to $D \operatorname{Tr} C$ and f is right almost split.*

is simple and equal to the $\mathrm{End}(D\,\mathrm{Tr}\,C)$-socle follows by duality. Since in each of the two socles the nonzero elements are the almost split sequences, the socles must coincide. □

We now proceed to give a series of characterizations of almost split sequences.

Proposition 2.2 *Let C be a nonprojective indecomposable Λ-module and $\delta: 0 \to D\,\mathrm{Tr}\,C \xrightarrow{f} B \xrightarrow{g} C \to 0$ a nonsplit exact sequence. Then the following are equivalent.*

(a) *δ is an almost split sequence.*

(b) *Each nonisomorphism $h: C \to C$ factors through $g: B \to C$.*

(c) $\mathrm{Im}(\mathrm{Hom}_\Lambda(C, g)) = \mathrm{rad}(\mathrm{End}_\Lambda(C)^{\mathrm{op}})$.

(d) *$\delta^*(C)$ is a simple $\mathrm{End}_\Lambda(C)^{\mathrm{op}}$-module, where δ^* denotes the contravariant defect of δ.*

(e) *$\delta^*(X) = 0$ for each indecomposable module X which is not isomorphic to C.*

(f) *δ generates the socle of $\mathrm{Ext}^1_\Lambda(C, D\,\mathrm{Tr}\,C)$ as an $\mathrm{End}_\Lambda(C)^{\mathrm{op}}$-module.*

(b′) *Each nonisomorphism $h: D\,\mathrm{Tr}\,C \to D\,\mathrm{Tr}\,C$ factors through $f: D\,\mathrm{Tr}\,C \to B$.*

(c′) $\mathrm{Im}(\mathrm{Hom}_\Lambda(f, D\,\mathrm{Tr}\,C)) = \mathrm{rad}(\mathrm{End}_\Lambda(D\,\mathrm{Tr}\,C))$.

(d′) *$\delta_*(D\,\mathrm{Tr}\,C)$ is a simple $\mathrm{End}_\Lambda(D\,\mathrm{Tr}\,C)$-module, where δ_* is the covariant defect.*

(e′) *$\delta_*(X) = 0$ for each indecomposable module X which is not isomorphic to $D\,\mathrm{Tr}\,C$.*

(f′) *δ generates the socle of $\mathrm{Ext}^1_\Lambda(C, D\,\mathrm{Tr}\,C)$ as an $\mathrm{End}_\Lambda(D\,\mathrm{Tr}\,C)$-module.*

From Wikipedia

Suppose that R is an Artin algebra. A sequence

$0 \to A \to B \to C \to 0$

of finitely generated left modules over R is called an **almost - split sequence** (or **Auslander - Reiten sequence**) if it has the following properties:

- The sequence is not split.
- C is indecomposable and any homomorphism from an indecomposable module to C that is not an isomorphism factors through B .
- A is indecomposable and any homomorphism from A to an indecomposable module that is not an isomorphism factors through B.

Consequence

An Auslander - Reiten sequence

$$0 \longrightarrow \tau(\mathbf{M}) \xrightarrow{\mathbf{f}} X \xrightarrow{\mathbf{g}} \mathbf{M} \longrightarrow 0$$

is a short exact sequence which does **NOT** split, with **M** & τ(**M**) **indecomposable** s.t. any $L \longrightarrow$ **M** non split epi (resp. τ(**M**) $\longrightarrow$ L non split mono) factors through **g** (resp. **f**).

Consequence (cont.)

- Any possible “candidate” **factors** through **f** and **g** .
- Any possible “candidate” obtained from an Auslander- Reiten sequence (by means of non - split morphisms, pushout and pullback) **splits**.

My belief/experience: also small pieces may be useful to understand the whole.

E.g. : the 2 non - zero **maps** in an **AR - sequence** are more important than the **irreducible maps** (between indecomposable modules), the small ingredients they are made of, **but more complicated** to compute and/or guess.

Why guess ?

- ONLY A FEW IRREDUCIBLE maps
(between indecomp. modules) are well - know.

- Symmetric considerations + "topology"
(= shape of irred. maps in the area) + **???**
suggest the form of the maps
$\tau(M) \longrightarrow X$ and $X \longrightarrow M$

possible meaning of **???** = **intuition**

(4, 15) , (4 , 14) , (1 , 8) ,
(3, 10) , (4 , 11) :

The dimension types

$$(a , b) = (\dim V(1) , \dim V(2))$$

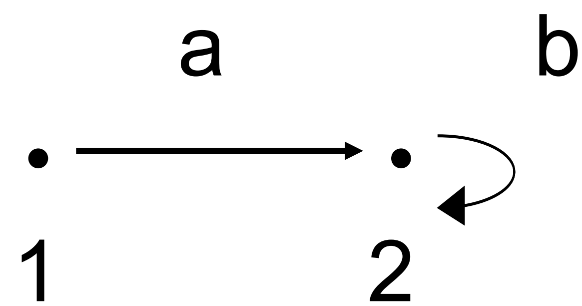
of the indecomp. modules **M** , **τ(M)** and of the 3 **indecomp. summands** of the middle term **?** in the AR - sequence

$$\mathbf{0} \longrightarrow \tau(\mathbf{M}) \longrightarrow \mathbf{?} \longrightarrow \mathbf{M} \longrightarrow \mathbf{0}$$

I want to show you.

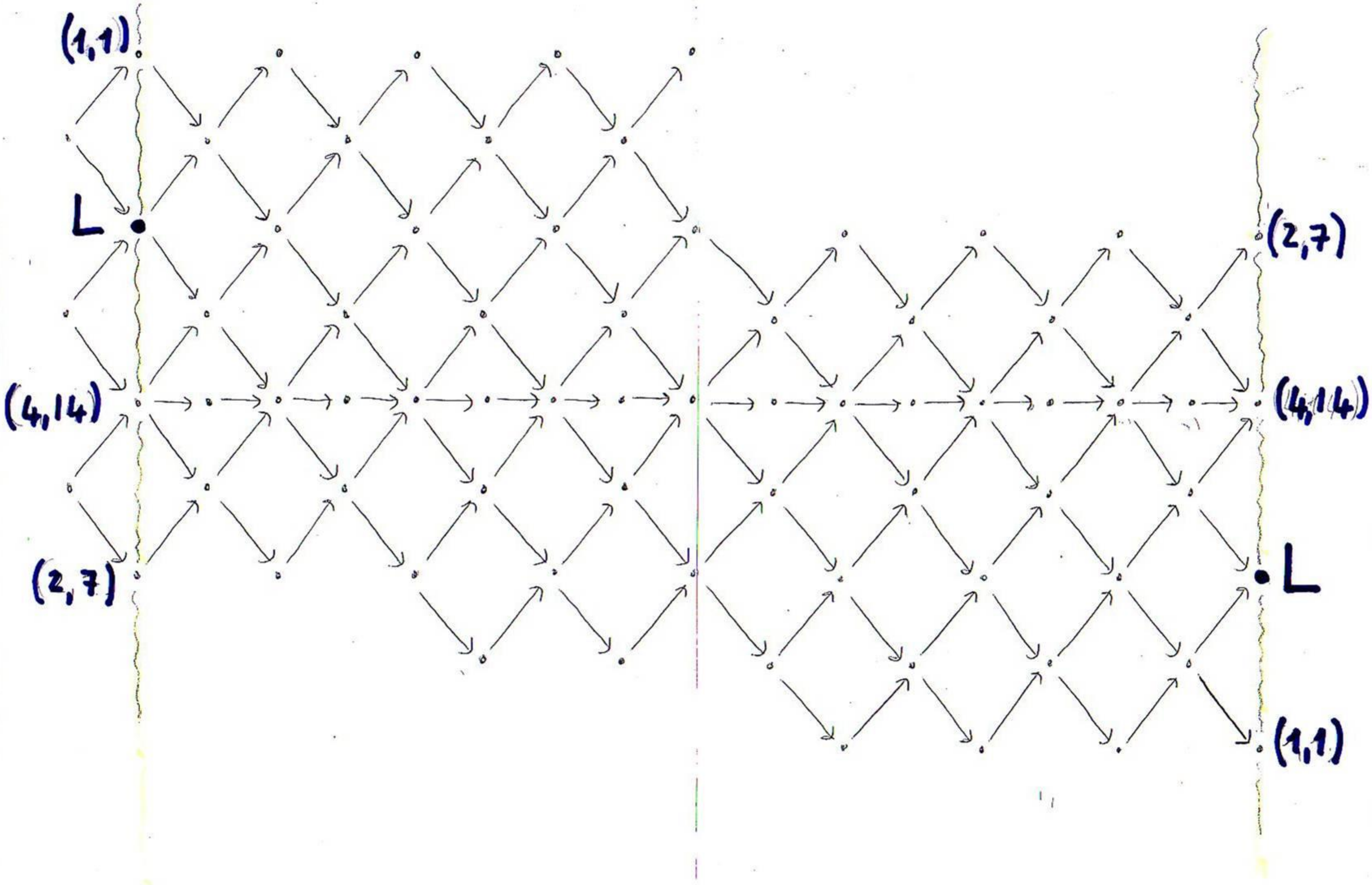
In this AR sequence the modules are defined over the algebra

given by the quiver



with $b^2 a = 0$ and $b^5 = 0$.

Its AR quiver is of the following form



(1,1)
L
(4,14)
(2,7)
(2,7)
(4,14)
L
(1,1)

As modules over $K[x] / (x^5)$, the vector spaces $V(2)$ in the indecomp. modules of dim. type (4,15) , …….. , (4,11) are direct sums of cyclic modules of length

2, 3, 5, 5 ; 2, 3, 4, 5 ;

3, 5 ; 2, 3, 5 ; 2 , 4, 5 .

The **injective** map $\boldsymbol{\tau(M)} \longrightarrow \mathbf{X}$

is of the form (f, g , h) with f, g and h

irreducible & surjective.

The **surjective** map $\mathbf{X} \longrightarrow \mathbf{M}$

is of the form (f , g, h) with f , g and h

irreducible & injective.

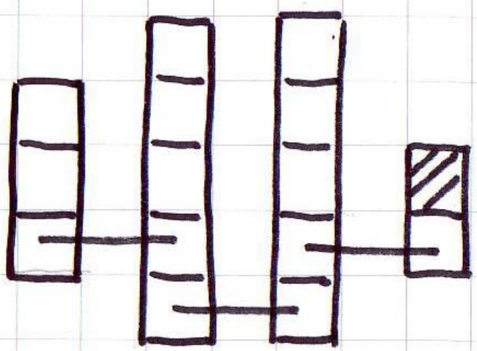
What happens:

With respect to suitable bases (actually 2) all irreducible morphisms are obvious maps ("additions" or "cancellations").

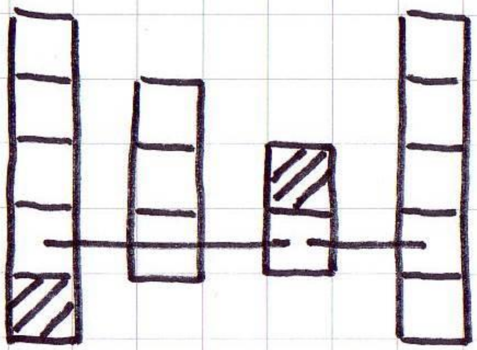
[Notation in the following:

$M = (4,15) = (4,15)\,A = (4,15)\,B$

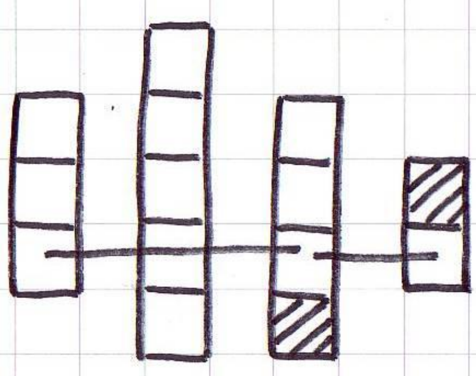
$\tau(M) = (4,14) = (4,14)\,A = (4,14)\,B$]



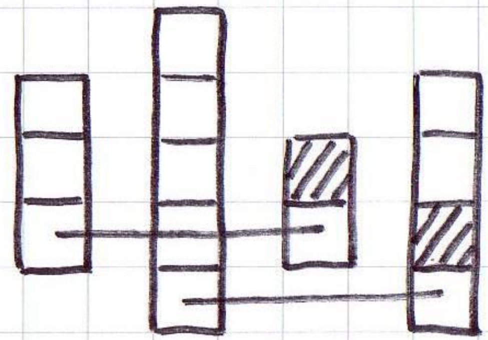
(4,15) A



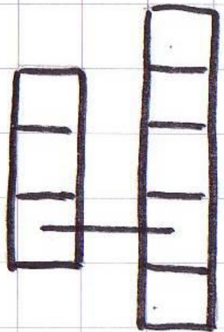
(4,15) B



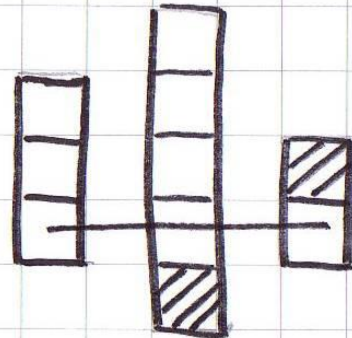
(4,14) A



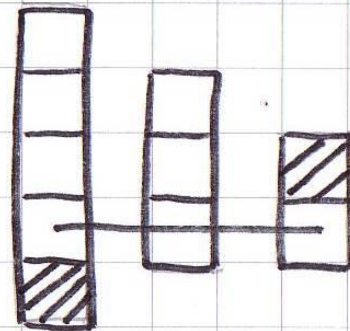
(4,14) B



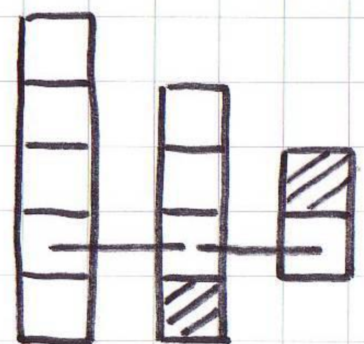
(1,8)



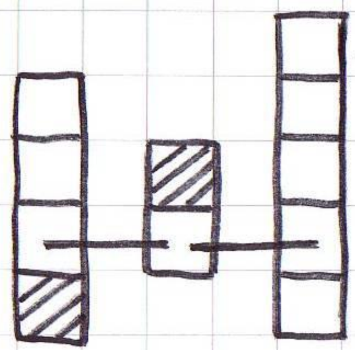
(3,10)A



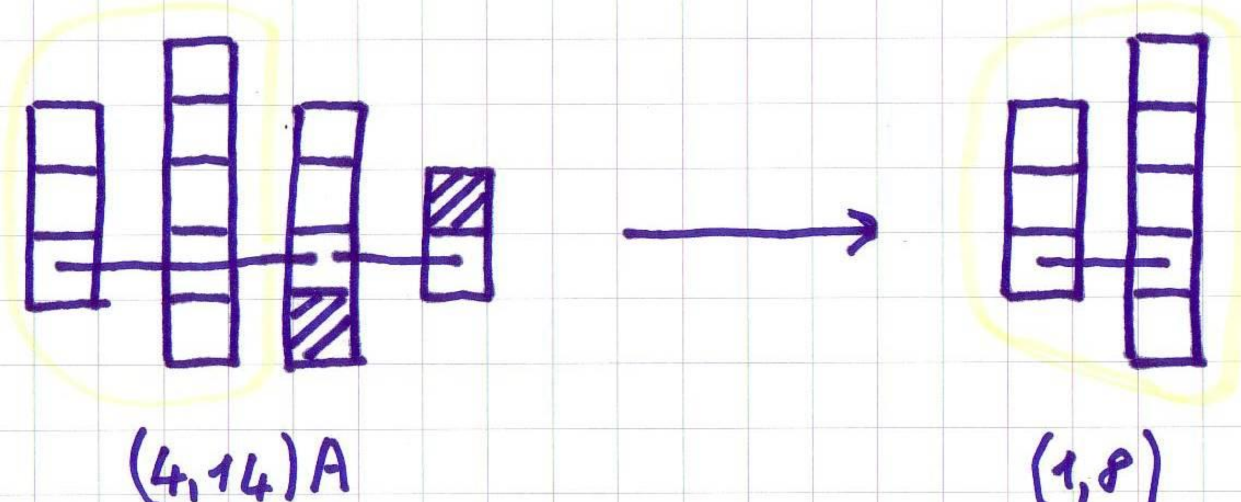
(3,10)B



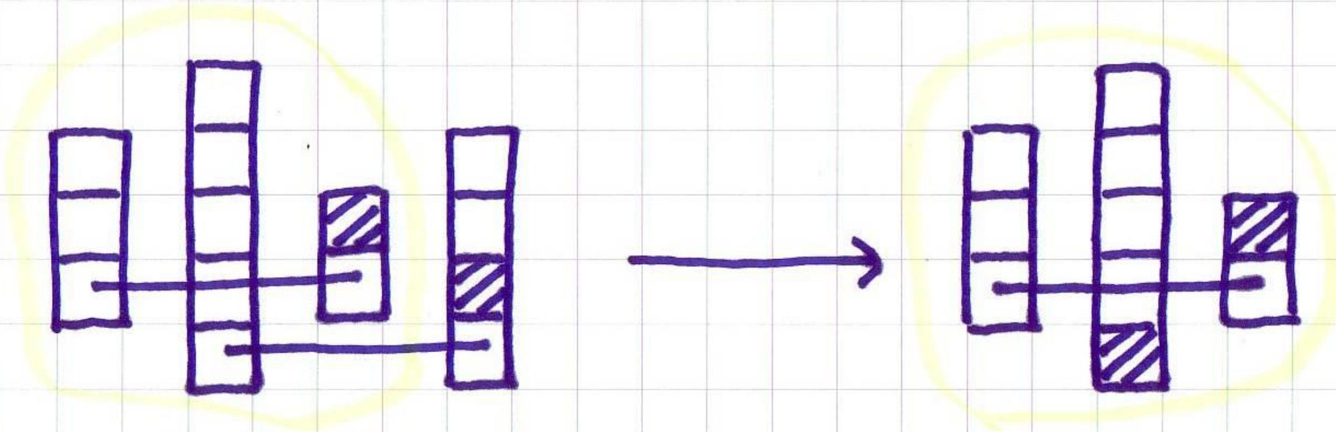
(4,11)A



(4,11)B

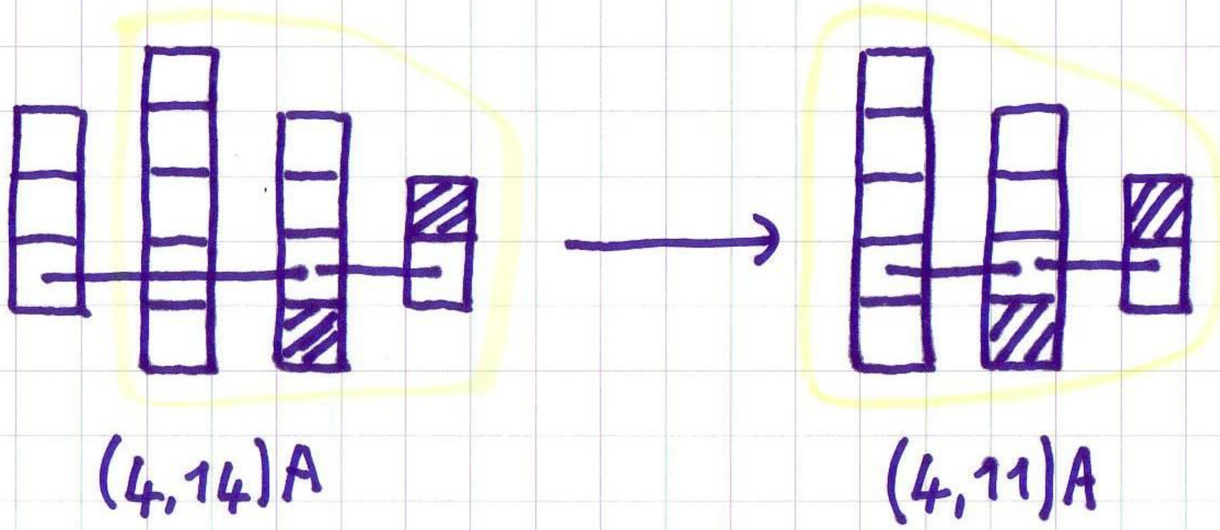


RIGHT CANCELLATION

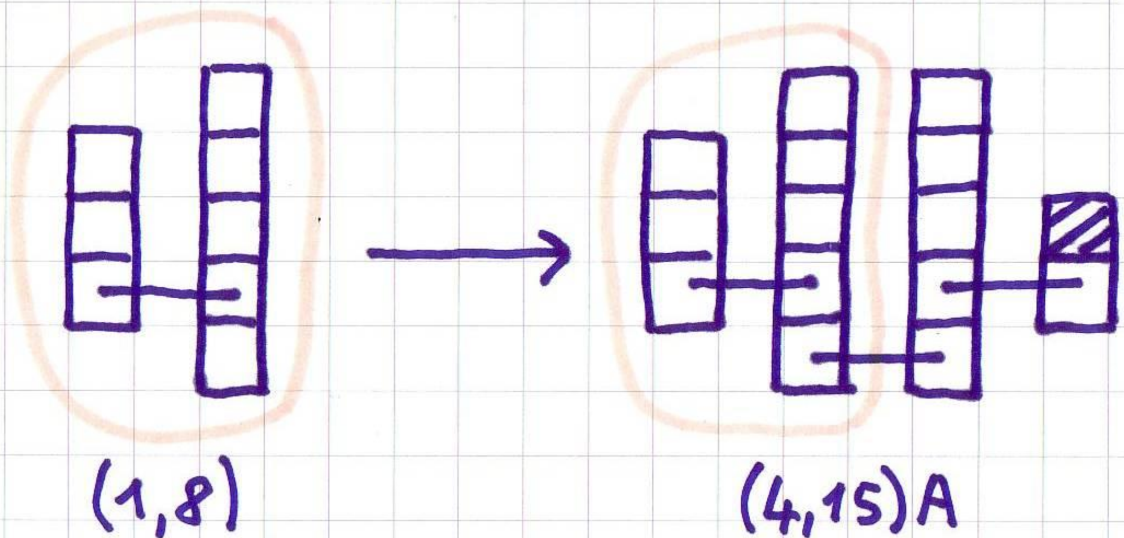


(4,14) B (3,10) A

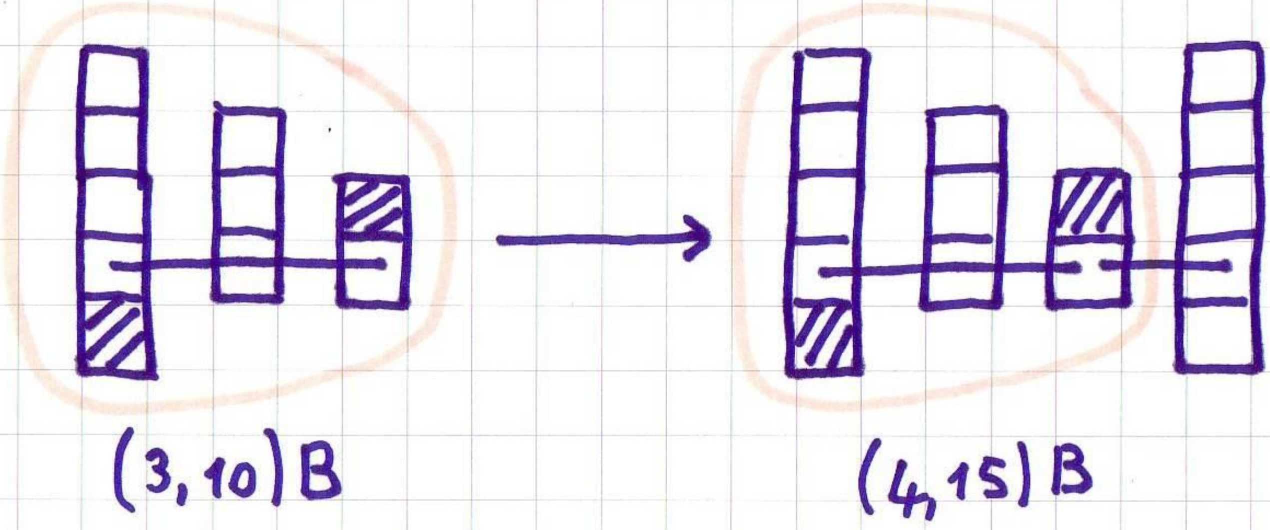
RIGHT CANCELLATION



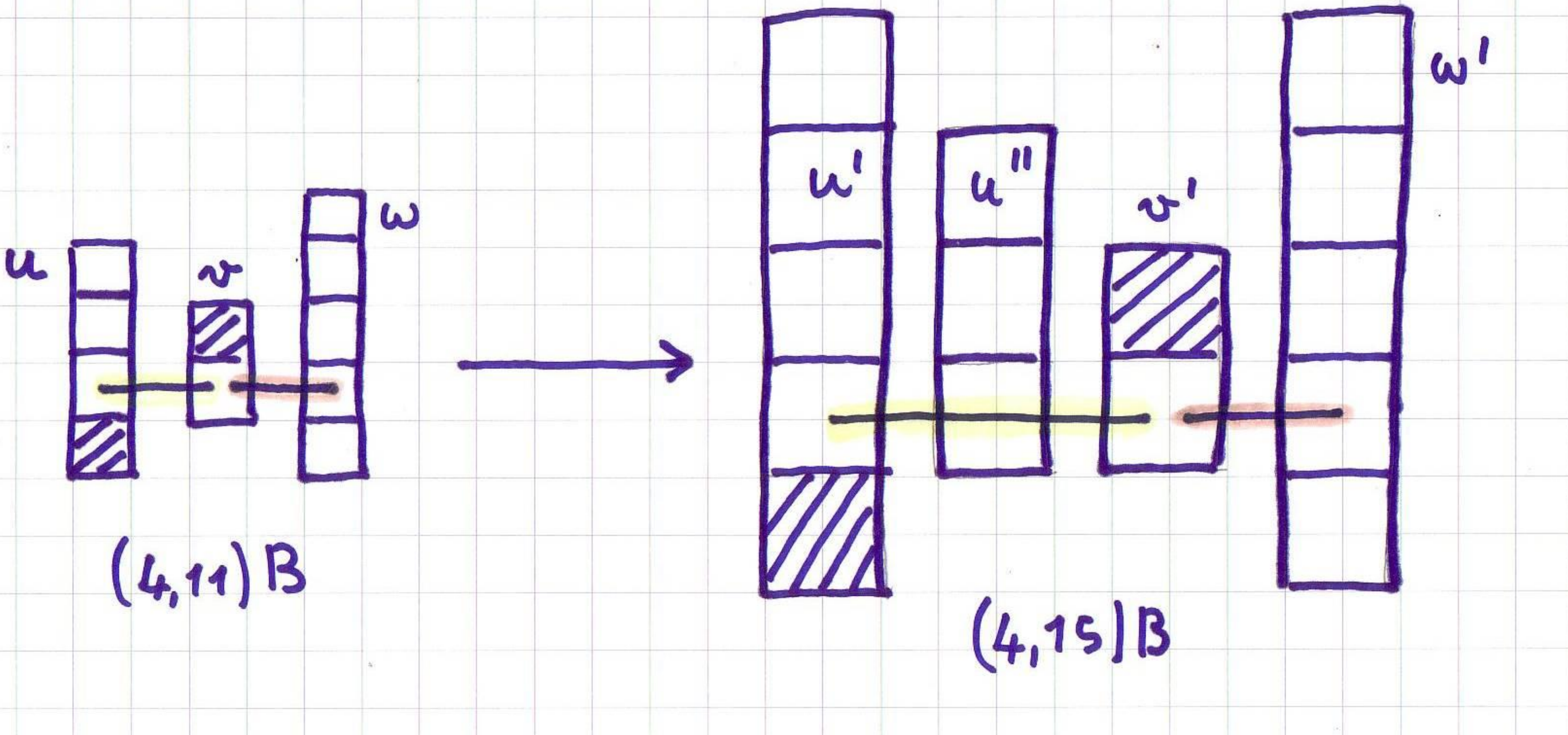
LEFT CANCELLATION



RIGHT ADDITION



RIGHT ADDITION



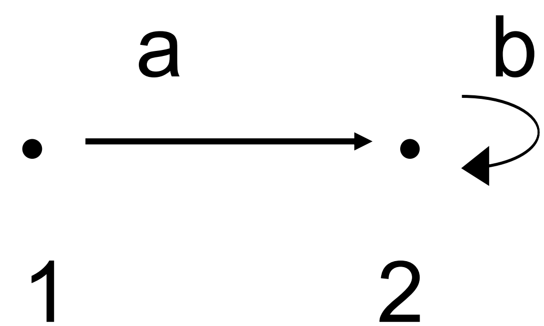
$u \mapsto u' + u''$, $v \mapsto v'$, $w \mapsto w'$

Possible choices for A and B in the next propositions

A

B

given by the quiver



with $b^2 a = 0$ and

$b^4 = 0$

$b^5 = 0$

Some properties of A and B :

- A and B have finite representation type;
- A admits **28** indecomposable modules (Ringel's paper and home page);
- B admits **66** indecomposable modules.
- If one replaces 4 or 5 by **2** or **3** , one obtains two algebras with **7** or **14** indecomposable modules.

Hypotheses in Proposition 1

A , B fin. dim algebras, $B \longrightarrow A$ surjective morphism;

$$s: \ 0 \longrightarrow M \xrightarrow{f} X \xrightarrow{g} \tau_A^{-}(M) \longrightarrow 0$$

AR - sequence of A - modules;

$$s^*: \ 0 \longrightarrow M \longrightarrow Y \longrightarrow \tau_B^{-}(M) \longrightarrow 0$$

AR - sequence of B - modules.

Proposition 1 : We may have

(a) $s = s^*$.
(b) X and Y indecomposable non isomorphic such that there is an epimorphism $F : Y \longrightarrow X$ with Ker F indecomposable and projective.
(c) X indecomposable and $Y = X \oplus P$ with P indecomposable and projective, f irreducible morphism of B - modules,

Proposition 1 (continuation)

g reducible morphism of B - modules.

(d) X decomposable, Y indecomposable such that there is an epimorphism $F : Y \longrightarrow X$ with Ker F indecomposable projective (resp. simple non projective).

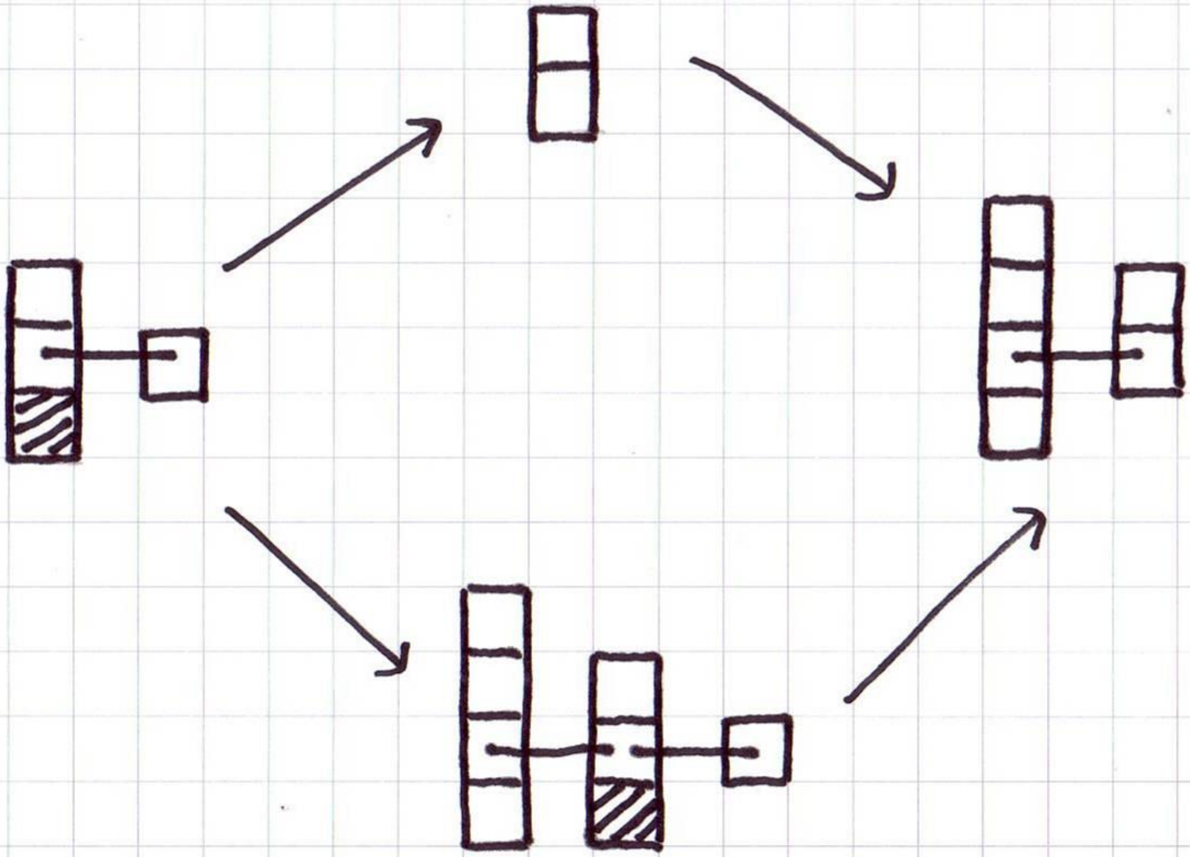
(e) f and g are irreducible (resp. reducible) maps of A - modules (resp. B - modules) and there is ....

Proposition 1 (continuation)

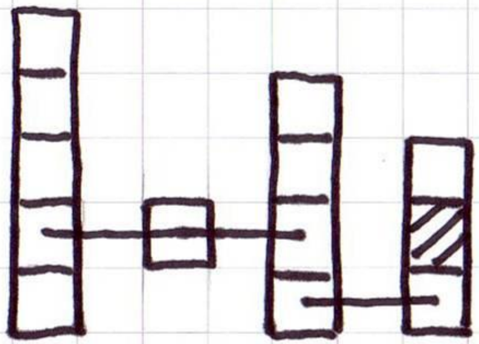
(f) f (resp. g) is the diagonal maps of 3 irreducible maps of A - modules; exactly 2 (resp. 1) of them are irreducible maps of B - modules, and there is an epimorphism $F: Y \longrightarrow X$ with Ker F indecomposable projective.

“Proof”

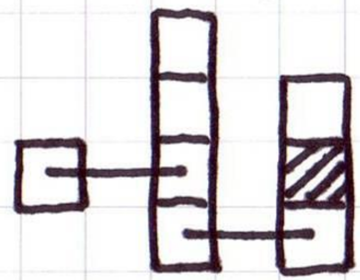
(a) AR sequence of Λ-modules (Λ = A, B)



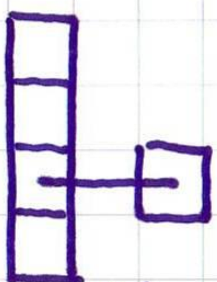
(b)



Y = (3, 13)

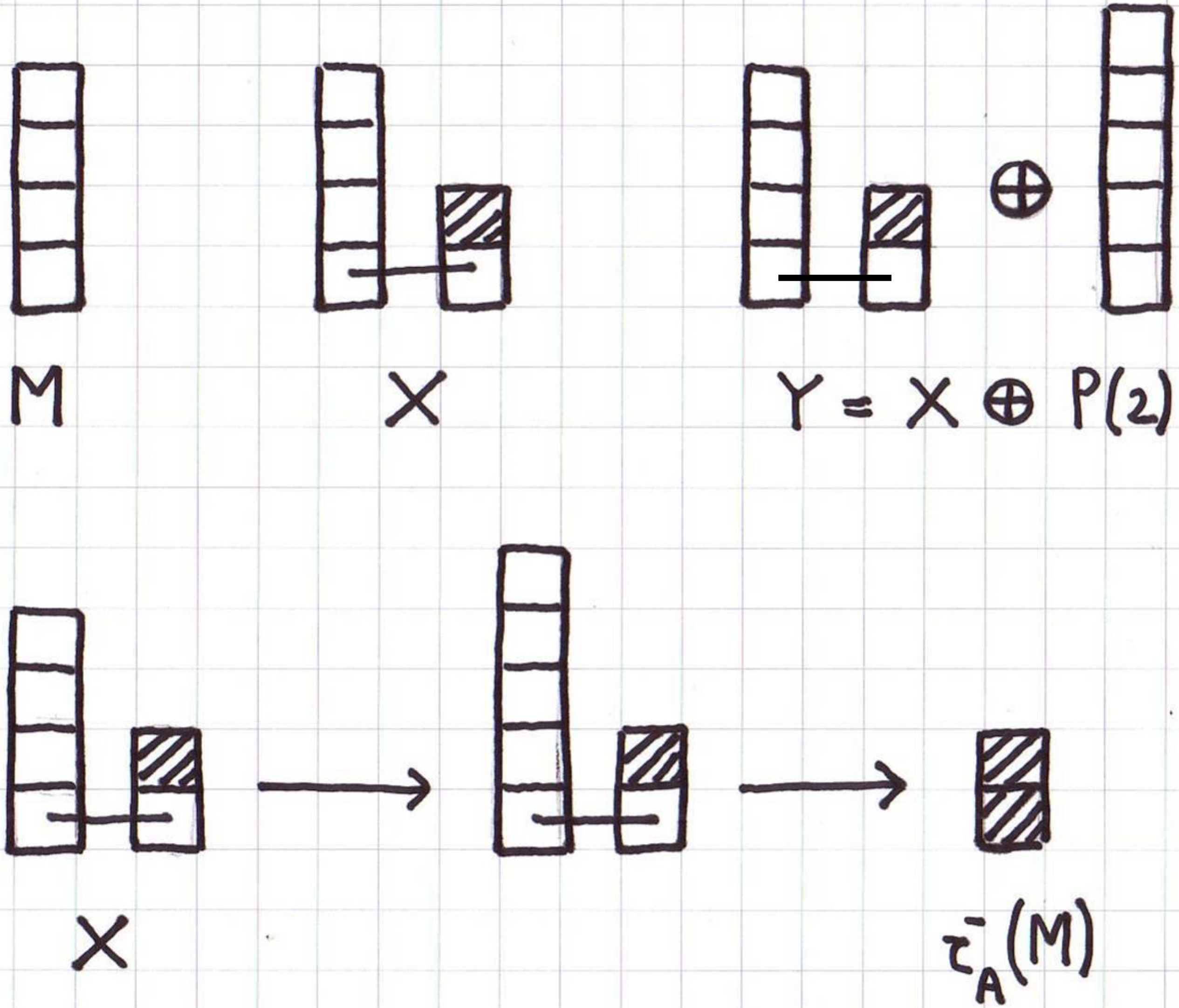


X = (3, 8)

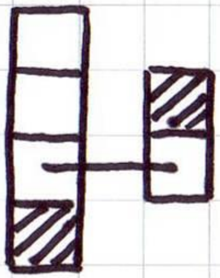


M

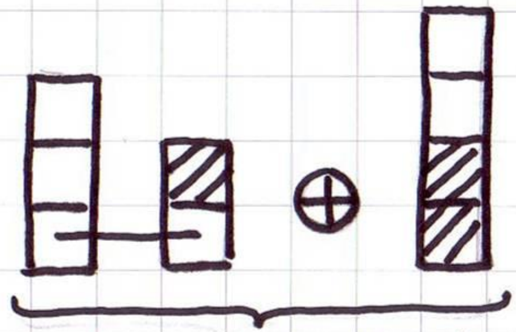
(c)



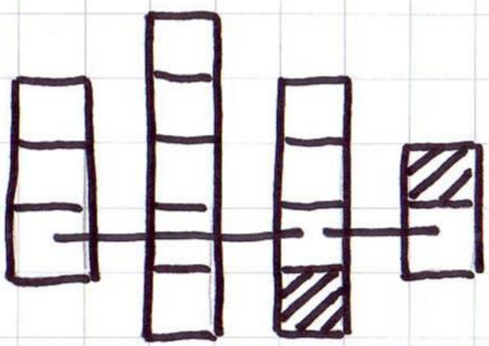
(d1)



M

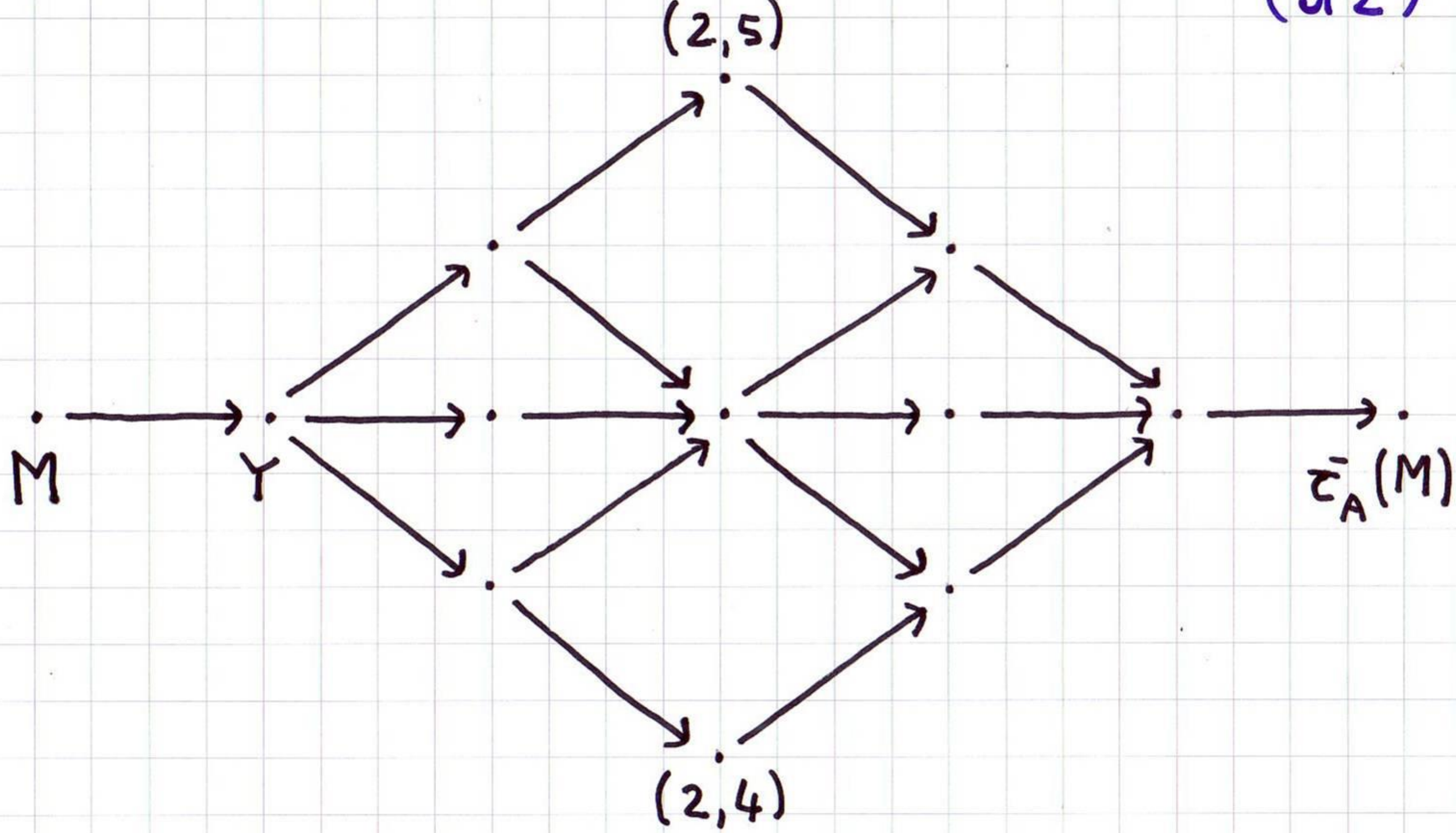


$X = (2,5) \oplus (2,4)$

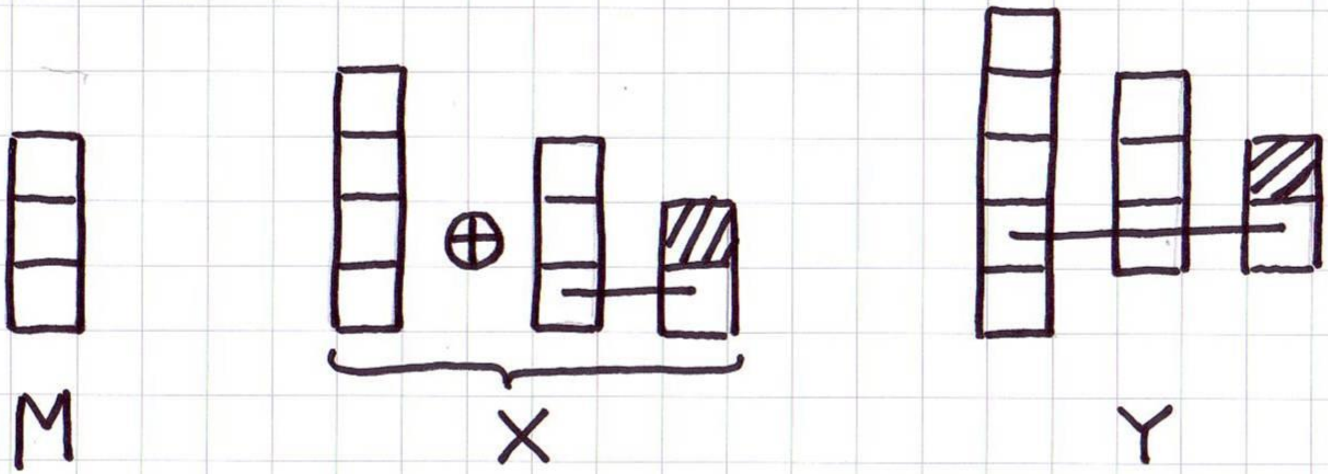


Y

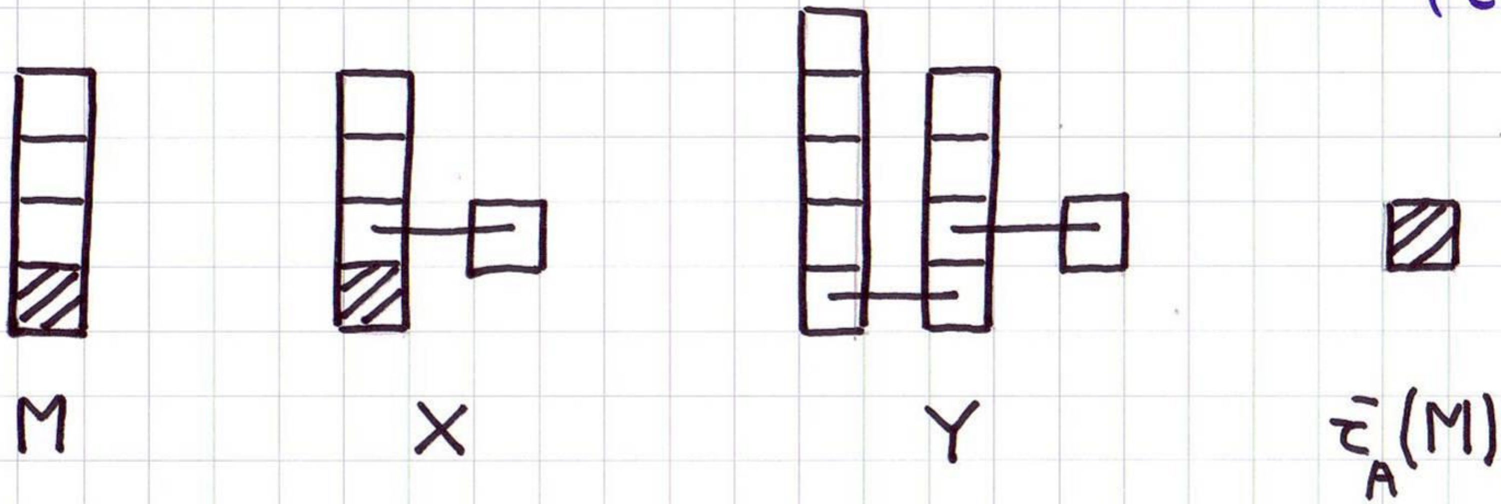
(d2)



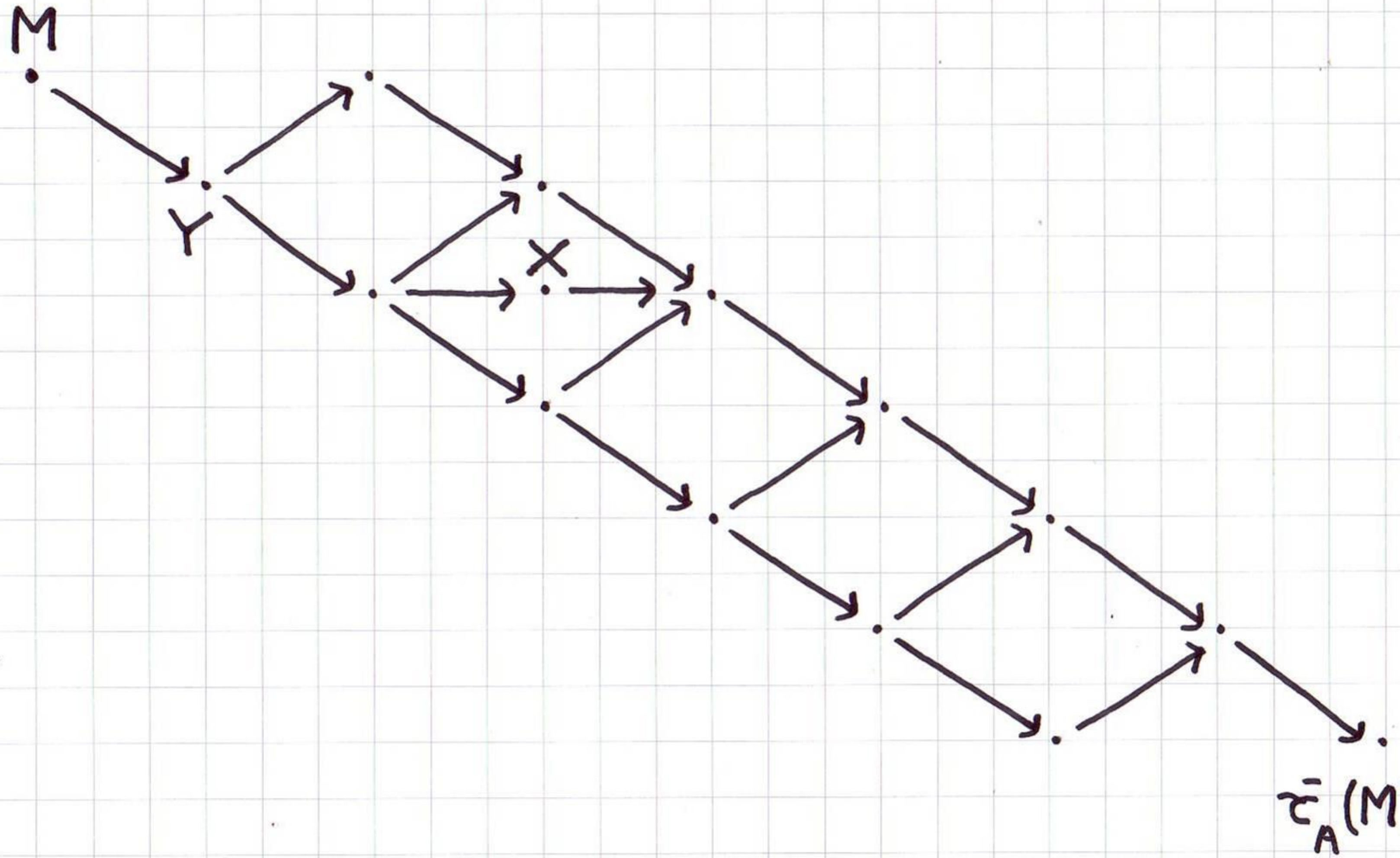
(d3)



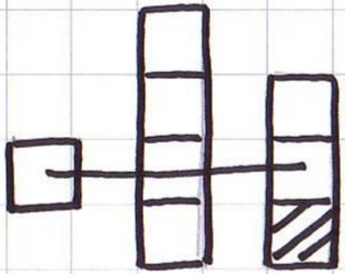
(e1)



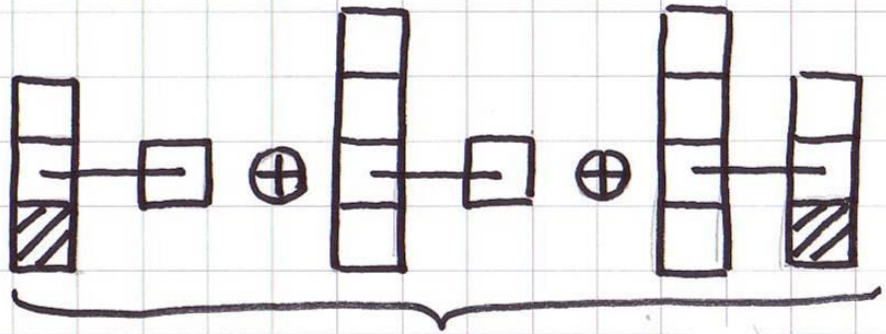
(e2)



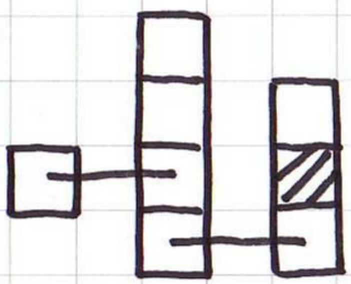
(f1)



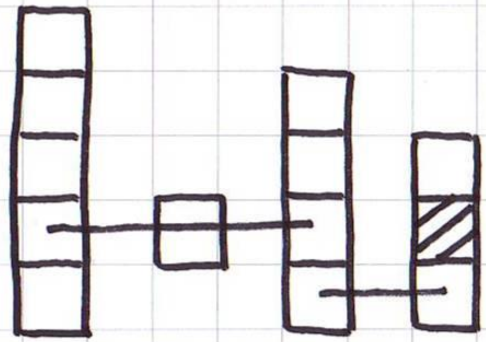
M



$X = (2,4) \oplus (1,5) \oplus (2,7)$

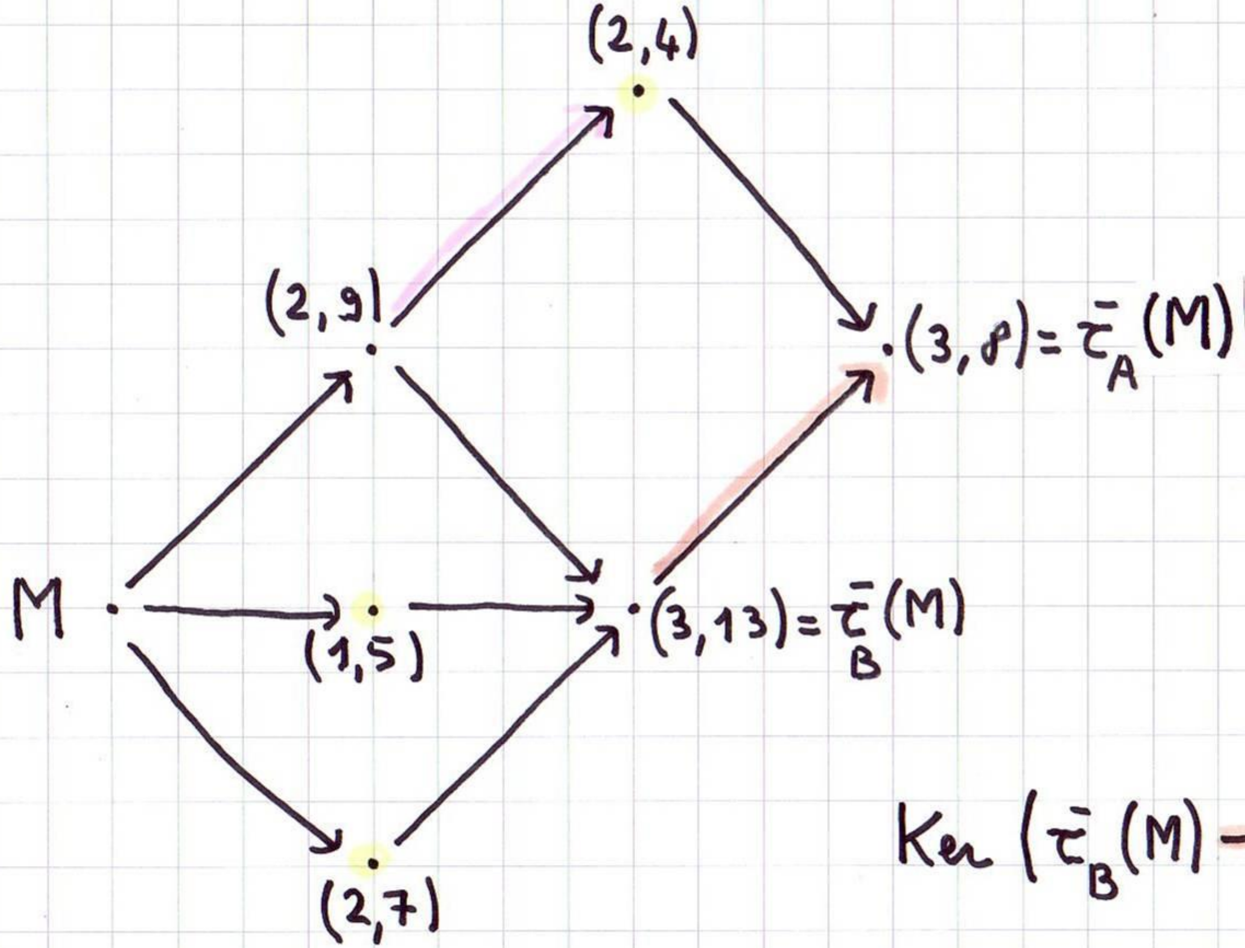


$\tau_A^-(M) = (3,8)$



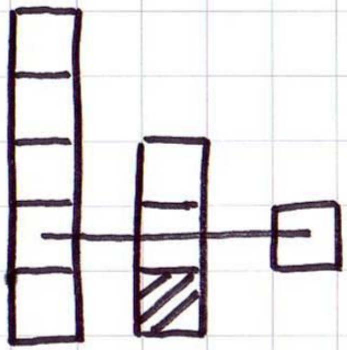
$\tau_B^-(M) = (3,13)$

(f2)

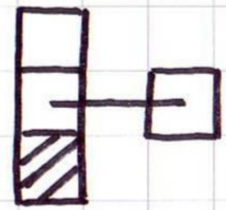


$$\mathrm{Ker}\,\left(\bar{\tau}_B(M) \longrightarrow \bar{\tau}_A(M)\right) = P(2)$$

(f3)



(2, 9)



(2, 4)

$$\mathrm{Ker}\big((2,9) \longrightarrow (2,4)\big) = P(2)$$

Hypotheses in Proposition 2

A , B fin. dim algebras, $B \longrightarrow A$ surjective;

$$s: \; 0 \longrightarrow \tau_A(M) \longrightarrow X \longrightarrow M \longrightarrow 0$$

AR - sequence of A - modules;

$$s^*: \; 0 \longrightarrow \tau_B(M) \longrightarrow Y \longrightarrow M \longrightarrow 0$$

AR - sequence of B - modules.

Proposition 2 : We may have

(i) X and Y are indecomposable non isomorphic modules and X is a maximal submodule of Y.

(ii) X is indecomposable and Y is of the form $X \oplus I$ with I indec. injective.

(iii) X is decomposable, Y is indecomp. and X is a maximal submodule of Y.

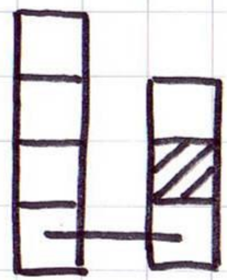
“Proof”

(i)

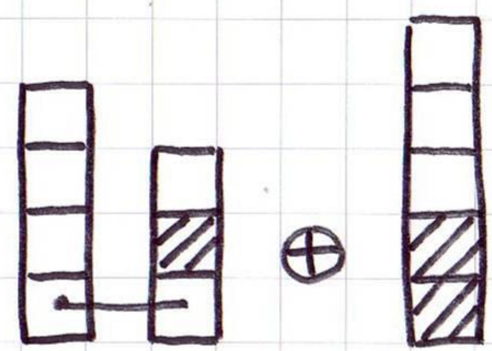
$M = I(1)$, $X =$  , $Y =$ 

ii

M

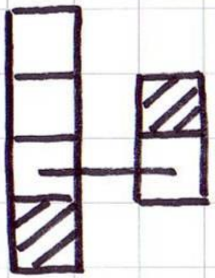


X

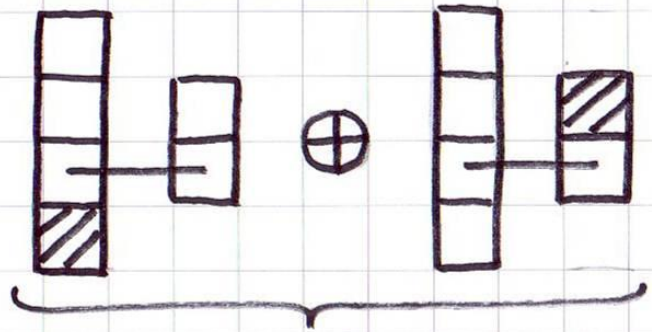


$Y = X \oplus I(2)$

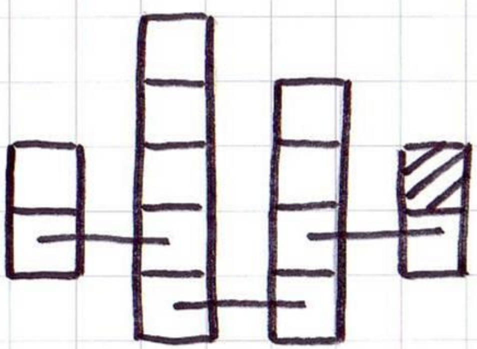
(iii)



M



X



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THANK YOU !