The Canonical Join Complex

Emily Barnard

North Carolina State University

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The canonical join representation

The **canonical join representation** of an element in a lattice (when it exists) is a unique minimal join-representation.

Theorem [Mizuno]

Suppose that W is a finite Weyl group, and Π is the corresponding preprojective algebra. There is a bijection between the torsion-free classes of Π and the elements of W that is an isomorphism of lattices.

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Theorem [Iyama, Reading, Reiten, and Thomas]

Suppose that Π is the preprojective algebra corresponding to the finite Weyl group W. Then the layers of the torsion-free class corresponding to w correspond to the canonical joinands of w.

Canonical Join Representation

Definition

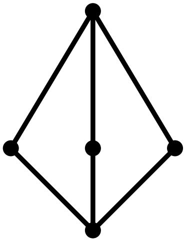
The **canonical join representation** of w (when it exists) is the unique minimal join representation $w = \bigvee A$ in the following sense:

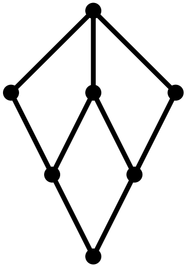
1. A is minimal in containment order, i.e. $\bigvee A$ is **irredundant**.
2. A is the minimal antichain in join-refinement order.

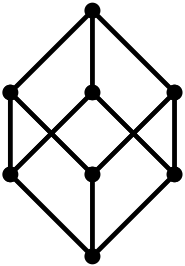
Easy

Each $j \in A$ is join-irreducible, and called a **canonical joinand of** w.

Examples







Examples

Definitions and Fact

- If each element of L has a CJR, then L is **join-semidistributive**.
- Dually, if each element of L has a canonical meet representation, then L is **meet-semidistributive**.
- If L is join-semidistributive, then every irredundant join of atoms is a CJR.

Question

Which subsets of join-irreducible elements form a CJR?

The canonical join complex

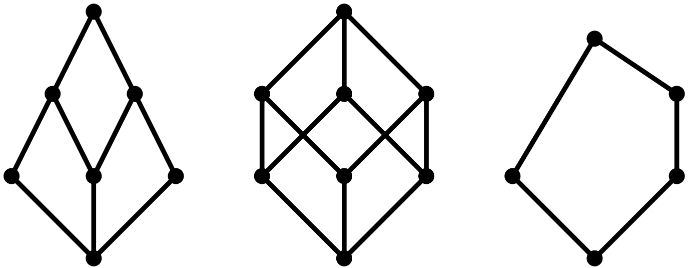
Proposition [Reading]

Suppose that $\bigvee A = w$ is a canonical join representation. Then the join $\bigvee A'$ for each subset $A' \subseteq A$ is also a canonical join representation.

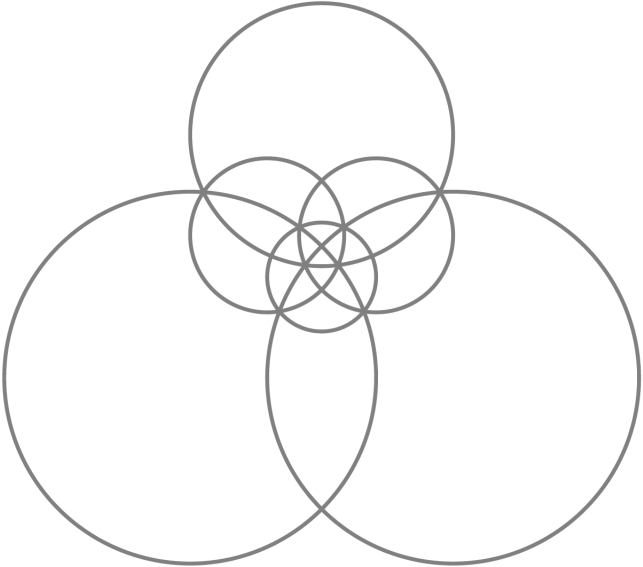
Definition

Suppose that L is a finite join-semidistributive lattice.
The **canonical join complex** of L is the simplicial complex whose faces are the sets of join-irreducible elements that join canonically.

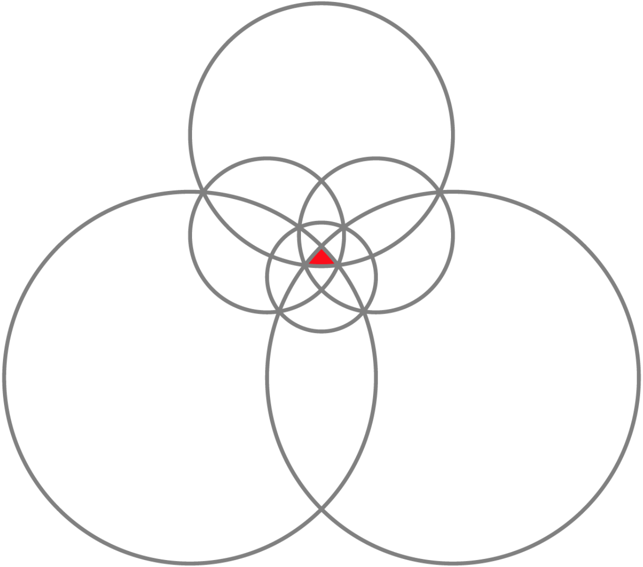
Examples



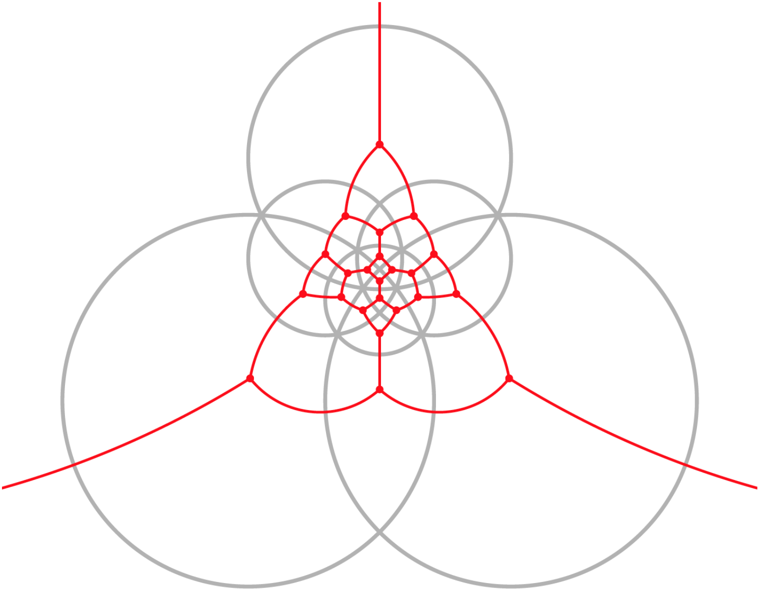
Poset of Regions and Shards



Poset of Regions and Shards



Poset of Regions and Shards



Poset of Regions and Shards

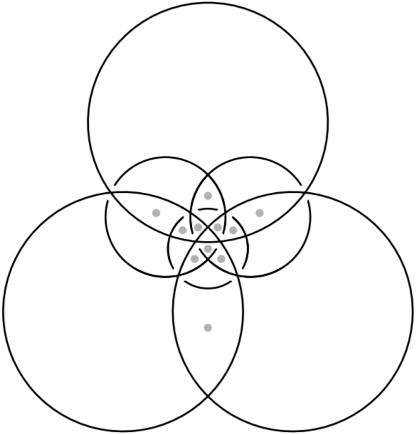
Facts

- If each region is a simplicial cone, then the poset of regions is a join-semidistributive lattice.
- The join-irreducible elements correspond to the **shards** of the arrangement.
- Shards are defined by 'cutting' hyperplanes in a way that encodes information about the lattice quotients of the poset of regions.

Poset of Regions and Shards

Fact

A pair of join-irreducible elements form a canonical join representation if and only if the corresponding shards intersect in their interiors.



The Weak order for a finite Coxeter Group

Definition

A **Coxeter system** (W, S) is a group with the presentation:

$$W = \langle s \in S : (ss')^{m(s,s')} = e \rangle$$

- $m(s, s') = m(s', s)$ and $m(s, s) = 1$ for each $s, s' \in S$.
- Let c denote a Coxeter element for (W, S).

The Weak order for a finite Coxeter Group

- The weak order for a finite Coxeter group W can be realized as a lattice of regions.
- Since each region is a simplicial cone, the weak order is a join-semidistributive lattice.

Question

What does the canonical join complex look like?

The Weak order for a finite Coxeter Group

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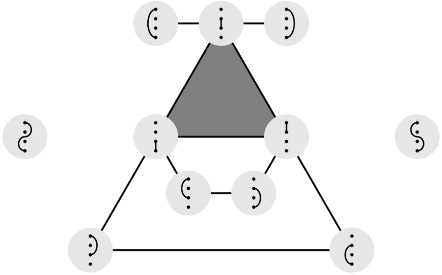

- In type A, shards can be represented by **noncrossing diagrams**, which encode the defining inequalities of the shard.

Main Results

Theorem [Reading]

The canonical join complex for A_n is flag.

Question

Is the canonical join complex always flag?

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Theorem [B.]

Suppose that L is a finite join-semidistributive lattice. The canonical join complex of L is flag if and only if L is also meet-semidistributive.

Consequences

Corollaries

- If the canonical join complex of L is flag, then every sublattice and quotient lattice of L also has a flag canonical join complex.
- If $\mathcal{A}$ is simplicial, then the canonical join complex of the lattice regions for $\mathcal{A}$ is flag.
- The canonical join complex for the weak order is flag.

Topological Results

The Tamari lattice is isomorphic to the lattice of torsion-free classes of the hereditary algebra for an equi-oriented type A quiver.

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Theorem [B.]

1. The canonical join complex for the Tamari lattice is either contractible or homotopy equivalent to the wedge of Catalan-many spheres all of the same dimension.
2. The canonical join complex for the type B Tamari lattice is homotopy equivalent to the wedge of type B Catalan-many spheres all of the same dimension.