Maximal rigid objects and cluster-tilting objects in 2-CY categories

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Notation

k: an algebraically closed field
$\mathcal{C}$: a k-linear Krull-Schmidt Hom-finite triangulated category
[1]: the shift functor of $\mathcal{C}$
$\mathrm{Ext}^n(X, Y) := \mathrm{Hom}(X, Y[n])$ for $X, Y \in \mathcal{C}$
add T: the additive subcategory of summands of finite sums of copies of T, for an object $T \in \mathcal{C}$.

For a subcategory $\mathcal{X}$ of $\mathcal{C}$, We put

$$\mathcal{X}^{\perp} := \{M \in \mathcal{C} | Hom(\mathcal{X}, M) = 0\}$$

$$^{\perp}\mathcal{X} := \{M \in \mathcal{C} | Hom(M, \mathcal{X}) = 0\}$$

Introduction

Definition

Let $\mathcal{C}$ be a triangulated category, a k-linear autofunctor $\mathbb{S} : \mathcal{C} \to \mathcal{C}$ is called a **Serre functor** of $\mathcal{C}$ if there exist functorial isomorphisms

$$Hom(X, Y) \cong D\, Hom(Y, \mathbb{S}X)$$

for any $X, Y \in \mathcal{C}$, where $D = \mathrm{Hom}_k(-, k)$ is the duality of k-spaces. The category $\mathcal{C}$ is called n-**Calabi-Yau** ($n \in \mathbb{Z}$), n-**CY** for short, if $\mathbb{S} = [n]$.

Remark

For a triangulated 2-CY category $\mathcal{C}$ there exist functorial isomorphisms for $\forall X, Y \in \mathcal{C}$

$$\mathrm{Ext}^1(X, Y) = D\,\mathrm{Ext}^1(Y, X).$$

Example

- For a finite-dimensional hereditary k-algebra H, the associated cluster category $\mathcal{C}_H$ is 2-CY. [BMRRT]
- For the preprojective algebra Λ of a Dynkin quiver over k, the stable category $\underline{mod}_\Lambda$ of Λ is 2-CY.[AR]
- For a one-dimensional simple hypersurface singularity (R, m, k) over an algebraically closed field k of characteristic zero, the stable category of maximal Cohen-Macaulay modules $\underline{CM}(R)$ is 2-CY. [BIKR]

Definition

Let $\mathcal{T}$ be a subcategory of a triangulated 2-CY category $\mathcal{C}$.

- $\mathcal{T}$ is called **rigid** if $\mathrm{Ext}^1(\mathcal{T}, \mathcal{T}) = 0$.
- $\mathcal{T}$ is called **maximal rigid** if $\mathcal{T}$ is rigid and is maximal with respect to this property, i.e. if $\mathrm{Ext}^1(\mathcal{T} \cup \mathrm{add}\, M, \mathcal{T} \cup \mathrm{add}\, M) = 0$, then $M \in \mathcal{T}$.
- $\mathcal{T}$ is called **cluster-tilting** if $\mathcal{T}$ is functorially finite and $\mathcal{T} = \mathcal{T}[-1]^{\perp} = {}^{\perp}\mathcal{T}[1]$, i.e. $\mathcal{T}$ is functorially finite rigid and for $\forall M \in \mathcal{C}$ if $\mathrm{Ext}^1(\mathcal{T}, M) = 0$, then $M \in \mathcal{T}$.

An object T is called **rigid, maximal rigid** or **cluster-tilting** if $\mathrm{add}\, T$ is rigid, maximal rigid or cluster-tilting subcategory respectively.

Introduction

Remark

By definition, it is easy to see that cluster-tilting subcategories are functorially finite maximal rigid subcategories. But the converse is not true in general [BIKR, BMV].

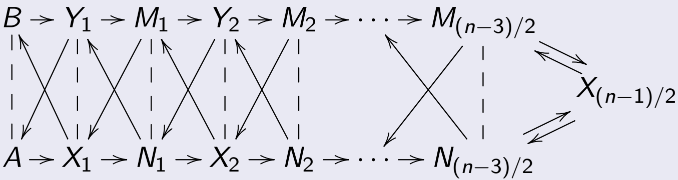
Example

In [BMV], Buan, Marsh and Vatne observed that the cluster tubes contain maximal rigid objects, but none of them are cluster-tilting objects.

Example

Example ([BIKR])

Let (R, m, k) be a one-dimensional simple hypersurface singularity where k is is an algebraically closed field of characteristic 0. Then the stable category $\underline{CM}(R)$ is a triangulated 2-CY category [BIKR]. In the case D_n with n odd, the AR-quiver of $\underline{CM}(R)$ is the following ([BIKR] or [Y])

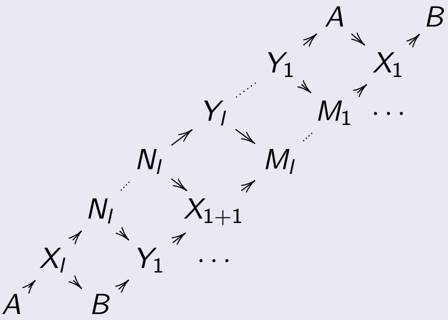


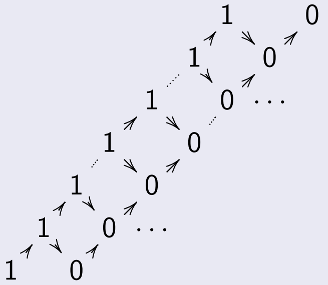
where the dotted line between two indecomposable modules means that they are connected via τ.

Example

Example

It was proved in [BIKR] that the indecomposable modules A and B are maximal rigid objects in $\underline{CM}(R)$ but none of them is cluster-tilting. And the computation of the support of $\underline{Hom}(A, -)$ is the following where $B = \tau A$ and $l = (n-3)/2$.





Hence the quiver of $\underline{End}(A)$ has a loop, the case of B is the same.

Buan, Iyama, Reiten and Scott proposed a conjecture in [BIRS].

Conjecture (Conjecture II.1.9 [BIRS])

Let $\mathcal{C}$ be a connected Hom-finite triangulated 2-CY category. Then any maximal rigid object without loops or 2-cycles in its quiver is a cluster-tilting object.

We proved the conjecture in the case of that the AR-quiver of $\mathcal{C}$ is connected.

Theorem ([XO])

Let $\mathcal{C}$ be a Hom-finite triangulated 2-CY category. If the AR-quiver of $\mathcal{C}$ is connected, then any maximal rigid object without loops in its quiver is a cluster-tilting object.

Sketch of proof

From now on, we fix $\mathcal{C}$ to be a 2-CY category, and T a basic maximal rigid object of $\mathcal{C}$.

Lemma (Iyama and Yoshino in [IY])

Assume $T = T_i \oplus T_i'$ where T_i is indecomposable, put $\mathcal{D} := \mathrm{add} T_i'$ and $\mathcal{Z} := \mathcal{D}[-1]^{\perp} = {}^{\perp}\mathcal{D}[1]$,, then $\mathcal{U} := \mathcal{Z}/\mathcal{D}$ is also a triangulated 2-CY category, where the shift functor of $\mathcal{U}$ is denoted by $\langle 1 \rangle$.

Lemma

If the quiver of $End(T)$ has no loops, then the following non-split triangles

$$T\langle 1\rangle \longrightarrow 0 \longrightarrow T \longrightarrow T \ ,$$

$$T \longrightarrow 0 \longrightarrow T\langle 1\rangle \longrightarrow T\langle 1\rangle$$

are AR-triangles in $\mathcal{U}$ defined above.

Sketch of proof

Proposition

Let $\mathcal{C}$ be a Hom-finite triangulated 2-CY category, T a basic maximal rigid object without loops in its quiver, put $\overline{\mathcal{D}} = T[-1]^{\perp} = {}^{\perp}T[1]$. Then the following are equivalent.

(1) *T is a cluster-tilting object;*

(2) ${}^{\perp}T \cap \overline{\mathcal{D}} \cap T^{\perp} = 0$;

(3) ${}^{\perp}T \cap \overline{\mathcal{D}} = 0$ *or* $\overline{\mathcal{D}} \cap T^{\perp} = 0$.

Sketch of proof

Theorem ([XO])

Let $\mathcal{C}$ be a Hom-finite triangulated 2-CY category. If the AR-quiver of $\mathcal{C}$ is connected, then any maximal rigid object without loops in its quiver is a cluster-tilting object.

Sketch of proof: If there is an non-zero indecomposable object M in ${}^{\perp}T \cap \overline{\mathcal{D}} \cap T^{\perp}$, then we can prove the connected AR-component which contains M is also contained in ${}^{\perp}T \cap \overline{\mathcal{D}} \cap T^{\perp}$, which is of course contained in ${}^{\perp}T$. So T and M cannot be in the same connected component. Contradiction

Recent work

We have proved the conjecture, that is

Theorem

Let $\mathcal{C}$ be a connected Hom-finite triangulated 2-CY category. Then any maximal rigid object without loops in its quiver is a cluster-tilting object.

Remark

We do not use the condition that the quiver of $\mathrm{End}_{\mathcal{C}} T$ has no 2-cycles.

Proof

Idea: To prove it by checking $T^{\perp} \cap \overline{\mathcal{D}} \cap {}^{\perp} T = 0$ (by the Proposition).
We found the following key Lemma:

Lemma

$$T^{\perp} \cap \overline{\mathcal{D}} \cap {}^{\perp} T = T[1]^{\perp} \cap \overline{\mathcal{D}}[1] \cap {}^{\perp} T[1]$$

Proof

By the Lemma above we can separated the category $\mathcal{C}$ into two disconnected parts, i.e. the following Lemmas:

Lemma

For any indecomposable object $M \in \mathcal{C}$, we have $M \in T^{\perp} \cap \overline{\mathcal{D}} \cap^{\perp} T$ or $M \in T * T[1]$, where $T * T[1] := \{X \in \mathcal{C} |$ there is an exact triangle $T_1 \to T_0 \to X \to T_1[1], T_1, T_0 \in addT\}$.

Lemma

$\mathcal{C}(T^{\perp} \cap \overline{\mathcal{D}} \cap^{\perp} T, T * T[1]) = 0$ *and* $\mathcal{C}(T * T[1], T^{\perp} \cap \overline{\mathcal{D}} \cap^{\perp} T) = 0$

Then the conjecture is a corollary of the two Lemmas. Because if $T^{\perp} \cap \overline{\mathcal{D}} \cap^{\perp} T \neq 0$, we have $\mathcal{C} = T^{\perp} \cap \overline{\mathcal{D}} \cap^{\perp} T \oplus T * T[1]$. It contradict to the connectedness of $\mathcal{C}$.

Thank you very much!

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