Resonance varieties, Hilbert series and Chen ranks

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Overview

1. Cohomology jump loci
   - The resonance varieties
   - The characteristic varieties

2. Alexander Modules and Chen Lie algebras

3. McCool groups

4. Picture groups
The resonance varieties

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- Define a cochain complex of finite-dimensional $\mathbb{C}$-vector spaces,

$$(A, a) : A^0 \xrightarrow{a \cup -} A^1 \xrightarrow{a \cup -} A^2 \xrightarrow{a \cup -} \cdots,$$

with differentials given by left-multiplication by $a$.
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**Definition**

The *resonance varieties* of $G$ are the homogeneous subvarieties of $A^1$

$$R^i_d(G, \mathbb{C}) = \{ a \in A^1 \mid \dim_{\mathbb{C}} H^i(A; a) \geq d \} ,$$

defined for all integers $i \geq 1$ and $d \geq 1$. 
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**Definition**

The *resonance varieties* of $G$ are the homogeneous subvarieties of $A^1$

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defined for all integers $i \geq 1$ and $d \geq 1$.

- $\mathcal{R}_1^1(\mathbb{Z}^n, \mathbb{C}) = \{0\}; \mathcal{R}_1^1(\pi_1(\Sigma_g), \mathbb{C}) = \mathbb{C}^{2g}, \ g \geq 2.$
The characteristic varieties

- $X$: connected CW-complex of finite type.
- $G = \pi_1(X)$. 

\[ T(X) := \text{Hom}(G, \mathbb{C}^\ast) = \text{Hom}(G_{ab}, \mathbb{C}^\ast) \] is an algebraic group, with multiplication 

\[ f_1 \circ f_2(g) = f_1(g) f_2(g) \] and identity \( id(g) = 1 \) for \( g \in G \) and \( f_i \in \text{Hom}(G, \mathbb{C}^\ast) \).

The rank 1 local system on \( X \) is a 1-dimensional \( \mathbb{C} \)-vector space \( C^\rho \) with a right \( C^G \)-module structure \( C^\rho \times G \rightarrow C^\rho \) given by \( \rho(g) \cdot a \) for \( a \in C^\rho \) and \( g \in G \) for \( \rho \in T(X) \).

**Definition**

The characteristic varieties of \( X \) over \( C \) are the Zariski closed subsets 

\[ V_i^d(X, C) = \{ \rho \in T(X) = \text{Hom}(G, \mathbb{C}^\ast) | \dim \mathbb{C}H_i(X, C^\rho) \geq d \} \] for \( i \geq 1 \) and \( d \geq 1 \).

\[ V_1^1(T_n, C) = \{ 1 \}; \quad V_1^1(\Sigma_g, C) = (\mathbb{C}^\ast)^{2g} \text{ for } g \geq 2. \]
The characteristic varieties

- $X$: connected CW-complex of finite type.
- $G = \pi_1(X)$.
- The character variety $\mathbb{T}(X) := \text{Hom}(G, \mathbb{C}^*) = \text{Hom}(G_{ab}, \mathbb{C}^*)$ is an algebraic group, with multiplication $f_1 \circ f_2(g) = f_1(g)f_2(g)$ and identity $\text{id}(g) = 1$ for $g \in G$ and $f_i \in \text{Hom}(G, \mathbb{C}^*)$. 
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The **characteristic varieties** of \( X \) over \( \mathbb{C} \) are the Zariski closed subsets

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Definition

The characteristic varieties of $X$ over $\mathbb{C}$ are the Zariski closed subsets

$$\mathcal{V}_d^i(X, \mathbb{C}) = \{ \rho \in \mathbb{T}(X) = \text{Hom}(G, \mathbb{C}^*) \mid \dim_{\mathbb{C}} H_i(X, \mathbb{C}_\rho) \geq d \}$$

for $i \geq 1$ and $d \geq 1$.

- $\mathcal{V}_1^1(T^n, \mathbb{C}) = \{1\}$; $\mathcal{V}_1^1(\Sigma_g, \mathbb{C}) = (\mathbb{C}^*)^{2g}$ for $g \geq 2$. 
Tangent Cone Theorem

Theorem (Dimca, Papadima, Suciu 09)

If $G$ is 1-formal, then the tangent cone $\text{TC}_1(\mathcal{V}_d^1(G, \mathbb{C}))$ equals $\mathcal{R}_d^1(G, \mathbb{C})$. Moreover, $\mathcal{R}_d^1(G, \mathbb{C})$ is a union of rationally defined linear subspaces of $H^1(G, \mathbb{C})$. 
Alexander invariant

- **Alexander invariant** is the $\mathbb{Z}[G_{ab}]$-module $B(G) = G'/G''$, where $G' = [G, G]$ and $G'' = [G', G']$ are the 1st and 2nd derived subgroups.
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- The $\mathbb{Z}[G_{ab}]$-module structure on $B(G)$ is determined by the extension

$$0 \to G'/G'' \to G/G'' \to G/G' \to 0.$$

with $G/G'$ acting on the cosets of $G''$ via conjugation: $gG' \cdot hG'' = ghg^{-1}G''$, for $g \in G$, $h \in G'$. 
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- The $i$-th **Fitting ideal** of a $\mathbb{C}[G_{ab}]$-module is the ideal in $\mathbb{C}[G_{ab}]$ generated by the co-dimension $i$ minors of the presentation matrix.
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**Proposition (Hironaka(97), Libgober(98) ...)**

\[
\mathcal{V}_d^1(G, \mathbb{C}) = V(E_{d-1}(B(G) \otimes \mathbb{C})) \text{ for } d \geq 1.
\]
Chen Lie algebras

- The *lower central series* of $G$: $\Gamma_1 G = G$, $\Gamma_2 G = G' = [G, G]$, $\Gamma_{k+1} G = [\Gamma_k G, G]$, $k \geq 1$. 

The associated graded Lie algebra of a group $G$ is defined to be $\text{gr}(G; C) := \bigoplus_{k \geq 1} (\Gamma_k(G) / \Gamma_{k+1}(G)) \otimes \mathbb{Z}C$.

The Chen Lie algebra of a group $G$ is defined to be $\text{gr}(G / G''; C) := \bigoplus_{k \geq 1} (\Gamma_k(G / G'') / \Gamma_{k+1}(G / G'')) \otimes \mathbb{Z}C$.

The quotient map $h: G \twoheadrightarrow G / G''$ induces $\text{gr}(G; C) \twoheadrightarrow \text{gr}(G / G''; C)$.

The LCS ranks of $G$ are defined as $\phi_k(G) := \text{rank}(\text{gr}_k(G; C))$.

The Chen ranks of $G$ are defined as $\theta_k(G) := \text{rank}(\text{gr}_k(G / G''; C))$.

$\theta_k(G) = \phi_k(G)$ for $k \leq 3$.

$\theta_k(F_n) = (k - 1)(n + k - 2)$, $k \geq 2$. [Chen (51)]
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Hilbert series and Chen ranks

\[ I := \ker \epsilon : \mathbb{Z}[G_{ab}] \rightarrow \mathbb{Z}. \]
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- \( l := \ker \epsilon : \mathbb{Z}[G_{ab}] \to \mathbb{Z} \).
- The module \( B(G) \) has an \( l \)-adic filtration \( \{ l^k B(G) \}_{k \geq 0} \).
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Proposition (Massey (80))

*For each \( k \geq 2 \), there exists an isomorphism*

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\text{gr}_k(G/G'') \cong \text{gr}_{k-2}(B(G)).
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**Corollary**

\[
\text{Hilb}(B(G) \otimes \mathbb{C}, t) = \sum_{k \geq 0} \theta_{k+2}(G) t^k.
\]
McCool groups (pure welded braid groups) (group of loops)

- The McCool group $wP_n$ is the group of basis-conjugating automorphisms, which is a subgroup of $IA_n := \ker(\text{Aut}(F_n) \to \text{GL}_n(\mathbb{Z}))$. 

$H^*(wP_n, \mathbb{C})$ was computed by Jensen, McCammond, and Meier (06).

Theorem (D. Cohen (09))

The first resonance variety of McCool group $wP_n$ is $R_{11}(wP_n, \mathbb{C}) = \bigcup 1 \leq i < j \leq n C_{ij} \cup \bigcup 1 \leq i < j < k \leq n C_{ijk}$, where $C_{ij} = C_2$ and $C_{ijk} = C_3$. 

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- The McCool groups $wP_n$ has a presentation [McCool (86)] with generators: $x_{ij}$, for $1 \leq i \neq j \leq n$ and relations: $x_{ij}x_{ik}x_{jk} = x_{jk}x_{ik}x_{ij}$; $[x_{ij}, x_{kl}] = 1$; $[x_{ij}, x_{kj}] = 1$, for $i, j, k, l$ distinct.
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**Theorem (D.Cohen (09))**

The first resonance variety of McCool group \( wP_n \) is

\[
\mathcal{R}_1^1(wP_n, \mathbb{C}) = \bigcup_{1 \leq i < j \leq n} C_{ij} \cup \bigcup_{1 \leq i < j < k \leq n} C_{ijk},
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where \( C_{ij} = \mathbb{C}^2 \) and \( C_{ijk} = \mathbb{C}^3 \).
Upper McCool groups

- The upper McCool group $wP_n^+$ is the subgroup of $wP_n$ generated by the $x_{ij}$ for $1 \leq i < j \leq n$. 

F. Cohen, Pakhianathan, Vershinin, and Wu (07) computed the cohomology ring $H^*(wP_n^+; \mathbb{Z})$. 

The LCS ranks $\varphi_k(wP_n^+)$ and the Betti numbers $b_k(wP_n^+)$, where $P_n$ is the pure braid group. 

They ask a question: are $wP_n^+$ and $P_n$ isomorphic for $n \geq 4$? For $n = 4$, the question was answered by Bardakov and Mikhailov (08) using Alexander polynomials. (There is a gap in the proof.)

Theorem (Suciu, W. (15)) The Chen ranks $\theta_k$ of $wP_n^+$ are given by

- $\theta_1 = \binom{n^2}{2}$,
- $\theta_2 = \binom{n^3}{3}$,
- $\theta_3 = 2 \binom{n+1^4}{4}$,
- $\theta_k = \binom{n+k-2}{k-2} + \theta_{k-1} = k \sum_{i=3}^{k} \binom{n+i-2}{i-2} + \binom{n+1^4}{4}$, $k \geq 4$.
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Upper McCool groups

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- F. Cohen, Pakhianathan, Vershinin, and Wu (07) computed the cohomology ring $H^*(wP_n^+; \mathbb{Z})$. The LCS ranks $\phi_k(wP_n^+) = \phi_k(P_n)$ and the Betti numbers $b_k(wP_n^+) = b_k(P_n)$, where $P_n$ is the pure braid group.
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**Theorem (Suciu, W. (15))**

The Chen ranks $\theta_k$ of $wP^+$ are given by $\theta_1 = \binom{n}{2}$, $\theta_2 = \binom{n}{3}$, $\theta_3 = 2\binom{n+1}{4}$, $\theta_k = \binom{n+k-2}{k+1} + \theta_{k-1} = \sum_{i=3}^{k} \binom{n+i-2}{i+1} + \binom{n+1}{4}$, $k \geq 4$. 

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Corollary

The pure braid group $P_n$, the upper McCool group $P\Sigma_n^+$, and the product group $\Pi_n := \prod_{i=1}^{n-1} F_i$ are not isomorphic for $n \geq 4$. 
**Corollary**

The pure braid group $P_n$, the upper McCool group $P\Sigma_n^+$, and the product group $\Pi_n := \prod_{i=1}^{n-1} F_i$ are *not* isomorphic for $n \geq 4$.

Proof:

\[\theta_4(P_n) = 3 \binom{n+1}{4}, \quad \theta_4(P\Sigma_n^+) = 2 \binom{n+1}{4} + \binom{n+2}{5}, \quad \theta_4(\Pi_n) = 3 \binom{n+2}{5}.\]

The Chen ranks of $P_n$ and $\Pi_n$ were computed by D. Cohen and Suciu (95).
The pure braid group $P_n$, the upper McCool group $P\Sigma_n^+$, and the product group $\Pi_n := \prod_{i=1}^{n-1} F_i$ are not isomorphic for $n \geq 4$.

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The Chen ranks of $P_n$ and $\Pi_n$ were computed by D. Cohen and Suciu (95).

Theorem (Suciu, W. (15))

The first resonance variety of upper McCool group $wP_n^+$ is
\[ R_1^{1}(wP_n^+, \mathbb{C}) = \bigcup_{1 \leq i < j \leq n-1} C_{i,j}, \]
where $C_{i,j} = \mathbb{C}^{j+1}$. 
Remark

There is a close connection (under some conditions) between the Chen ranks $\theta_k(G)$ and the resonance varieties $\mathcal{R}_1(G)$:

$$\theta_k(G) = \sum_{n \geq 2} c_m \cdot \theta_k(F_n), \quad \text{for} \quad k \gg 1,$$

where $c_n$ is the number of $n$-dimensional components of $\mathcal{R}_1(G)$.

(Suciu(01)) (Schenck and Suciu(04)) (D. Cohen and Schenck (14))

The pure braid groups $P_n$, the McCool groups $wP_n$, satisfy this formula. However, the upper McCool groups $wP_n^+$ does not satisfies this formula for $n \geq 4$. 

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Picture groups

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Picture groups

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- \(G(A_n)\): the picture group of type \(A_n\) with straight orientation, which is the fundamental group of the classifying space of the non-crossing category (Igusa (14)).
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- $G(A_n)$ is generated by $x_{ij}, (1 \leq i < j \leq n + 1)$, with relations
  \[
  \begin{cases}
  (x_{ij}, x_{kl}) = 1, & \text{if } (i, j), (k, l) \text{ are noncrossing}; \\
  (x_{ij}, x_{jk}) = x_{ik}, & \text{if } i < j < k,
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- $R(A_n) := \langle x_{i,i+1}, (1 \leq i \leq n) \mid (x_{i,i+1}, x_{j,j+1}) = 1, i < j - 1 \rangle$. 

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**Lemma**

There exists a surjection $R(A_n) \twoheadrightarrow G(A_n)$ inducing isomorphism on the resonance varieties $\mathcal{R}_d^1(G(A_n)) = \mathcal{R}_d^1(R(A_n))$. 
$R(A_n) := \langle x_{i,i+1}, (1 \leq i \leq n) \mid (x_{i,i+1}, x_{j,j+1}) = 1, i < j - 1 \rangle$ is a right-angled Artin group.
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All resonance varieties and characteristic varieties of right-angled Artin groups were computed by Papadima and Suciu (09). We only review the first resonance varieties here.

Theorem (Papadima-Suciu (06))

Let $\Gamma = (V, E)$ be a finite graph. Then $R_1(\Gamma; \mathbb{C}) = \bigcup W \mathbb{C} W$, where the union is over all subsets $W \subset V$ such that the induced subgraph $\Gamma_W$ is disconnected. Here, $\mathbb{C} W$ is the corresponding coordinate subspace of $\mathbb{C} V$. 

$R(A_3)$  

$R(A_4)$  

$R(A_5)$  

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• \( R(A_n) := \langle x_{i,i+1}, (1 \leq i \leq n) \mid (x_{i,i+1}, x_{j,j+1}) = 1, i < j - 1 \rangle \) is a right-angled Artin group.

![Diagrams of R(A3), R(A4), R(A5), R(A6)]

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Let \( \Gamma = (V, E) \) be a finite graph. Then \( \mathcal{R}_1(G_\Gamma; \mathbb{C}) = \bigcup_W \mathbb{C}^W \), where the union is over all subsets \( W \subset V \) such that the induced subgraph \( \Gamma_W \) is disconnected. Here, \( \mathbb{C}^W \) is the corresponding coordinate subspace of \( \mathbb{C}^V \).
Corollary

Recall that the graph corresponding to $R(A_n)$ has vertex set $\{x_{i,i+1}, (1 \leq i \leq n)\}$ and edges $(x_{i,i+1}, x_{j,j+1})$ for $i < j - 1$.

$$\mathcal{R}_1^1(G(A_n)) = \mathcal{R}_1^1(R(A_n)) = \bigcup_{i=1}^{n-2} \mathcal{C}^W_i$$

where $W_i = \{x_{i,i+1}, x_{i+1,i+2}, x_{i+2,i+3}\}$. 
Recall that the graph corresponding to $R(A_n)$ has vertex set \(\{x_{i,i+1}, (1 \leq i \leq n)\}\) and edges \((x_{i,i+1}, x_{j,j+1})\) for \(i < j - 1\).

\[
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\]

where \(W_i = \{x_{i,i+1}, x_{i+1,i+2}, x_{i+2,i+3}\}\).

\(R(A_3)\) \hspace{2cm} \(R(A_4)\) \hspace{2cm} \(R(A_5)\) \hspace{2cm} \(R(A_6)\)
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Recall that the graph corresponding to $R(A_n)$ has vertex set \{${x_i, i+1, (1 \leq i \leq n)}$\} and edges $(x_i, i+1, x_j, j+1)$ for $i < j - 1$.

$$\mathcal{R}_1^1(G(A_n)) = \mathcal{R}_1^1(R(A_n)) = \bigcup_{i=1}^{n-2} C W_i$$

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Example

$$\mathcal{R}_1^1(G(A_3)) = \mathbb{C}^3 = H^1(G(A_3); \mathbb{C}).$$

$$\mathcal{R}_1^1(G(A_4)) = \mathbb{C}^3 \cup \mathbb{C}^3 \subset H^1(G(A_4); \mathbb{C}) = \mathbb{C}^4.$$

$$\mathcal{R}_1^1(G(A_5)) = \mathbb{C}^3 \cup \mathbb{C}^3 \cup \mathbb{C}^3 \subset H^1(G(A_5); \mathbb{C}) = \mathbb{C}^5.$$

$$\mathcal{R}_1^1(G(A_6)) = \mathbb{C}^3 \cup \mathbb{C}^3 \cup \mathbb{C}^3 \cup \mathbb{C}^3 \subset H^1(G(A_6); \mathbb{C}) = \mathbb{C}^6.$$
Future work

- Compute the characteristic varieties of the picture group $G(A_n)$.

- Compute the Chen ranks of $G(A_n)$ and see the relations with the characteristic varieties.

- Investigate how these algebraic invariants reflect the information of the picture groups and the corresponding quivers.
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Thank You!