SILTING OBJECTS, t-STRUCTURES AND DERIVED EQUIVALENCES

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ABSTRACT. This note is an extended abstract of my talk given in the conference: "Maurice Auslander Distinguished Lectures and International Conference", April 29 - May 4, 2015. It is based on [6] which is joint work with Jorge Vitória.

CONTENTS

1. MOTIVATION AND PRELIMINARIES

Let R and S be two unital associative rings such that there is a triangle equivalence $\phi\colon \mathsf{D}^b(R) \xrightarrow{\simeq} \mathsf{D}^b(S)$ between the bounded derived categories of $\mathsf{Mod}\text{-}R$ and $\mathsf{Mod}\text{-}S$. Then it is known that the object $T := \phi(R)$ is a tilting complex in $\mathsf{D}^b(S)$ and that the ring R is the endomorphism ring $\mathsf{End}_{\mathsf{D}^b(S)}(T)$. Hence, via the equivalence ϕ we recover the module category of R via the module category of $\mathsf{End}_{\mathsf{D}^b(S)}(T)$. Another approach is to look where the standard t-structure generated by R is mapped via ϕ. For this reason, we recall from Beilinson-Bernstein-Deligne [4] the notion of a t-structure in triangulated categories.

Definition 1.1. [4] Let $\mathcal{T}$ be a triangulated category. A **t-structure** in $\mathcal{T}$ is a pair $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ of full subcategories such that, for $\mathcal{T}^{\leq n} = \Sigma^{-n}(\mathcal{T}^{\leq 0})$ and $\mathcal{T}^{\geq n} = \Sigma^{-n}(\mathcal{T}^{\geq 0})$ $(n \in \mathbb{Z})$, the following conditions are satisfied:

(i) $\mathsf{Hom}_{\mathcal{T}}(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 1}) = 0$, i.e. $\mathsf{Hom}_{\mathcal{T}}(X, Y) = 0$ for all X in $\mathcal{T}^{\leq 0}$ and Y in $\mathcal{T}^{\geq 1}$.
(ii) $\mathcal{T}^{\leq 0} \subseteq \mathcal{T}^{\leq 1}$ and $\mathcal{T}^{\geq 1} \subseteq \mathcal{T}^{\geq 0}$.
(iii) For every object D in $\mathcal{T}$ there exists a triangle $X \longrightarrow D \longrightarrow Y \longrightarrow \Sigma(X)$ such that X lies in $\mathcal{T}^{\leq 0}$ and Y lies in $\mathcal{T}^{\geq 1}$.

In the above situation we denote by $\tau^{\leq 0}(D) := X$ and $\tau^{\geq 1}(D) := Y$. Note that these objects are functorially determined. One of the main features of a t-structure is that it provides a full abelian subcategory (the heart) in $\mathcal{T}$ together with a cohomological functor. In particular, we have the next result.

Theorem 1.2. *[4] Let $\mathcal{T}$ be a triangulated category with a t-structure $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$. Then the* **heart** *$\mathcal{H} := \mathcal{T}^{\leq 0} \cap \mathcal{T}^{\geq 0}$ is an abelian category and there is a cohomological functor $H^0\colon \mathcal{T} \longrightarrow \mathcal{H}$ given by $H^0 = \tau^{\leq 0}\tau^{\geq 0} = \tau^{\geq 0}\tau^{\leq 0}$.*

Note that the exact structure of $\mathcal{H}$ is induced from $\mathcal{T}$ and there is an isomorphism $\mathsf{Ext}^1_{\mathcal{H}}(X, Y) \cong \mathsf{Hom}_{\mathcal{T}}(X, Y[1])$ for all $X, Y \in \mathcal{H}$.

Now we return to the discussion we had in the beginning of this section. For a ring R we have the standard t-structure:

$$\mathscr{D}_R^{\leq 0} = \{X \in \mathsf{D}^{\mathsf{b}}(R) \mid H^i(X) = 0 \ \forall i > 0\} \quad \text{and} \quad \mathscr{D}_R^{\geq 0} = \{X \in \mathsf{D}^{\mathsf{b}}(R) \mid H^i(X) = 0 \ \forall i < 0\}$$

in $\mathsf{D}^{\mathsf{b}}(R)$, where H^i is the usual cohomology of complexes. We leave to the reader to check that this is indeed a t-structure. Using that $H^i(X) = 0$ if and only if $\mathsf{Hom}_{\mathsf{D}^{\mathsf{b}}(R)}(R, X[i]) = 0$ we have

$$\mathscr{D}_R^{\leq 0} = R^{\perp_{>0}} := \{X \in \mathsf{D}^{\mathsf{b}}(R) \mid \mathsf{Hom}_{\mathsf{D}^{\mathsf{b}}(R)}(R, X[i]) = 0 \ \forall i > 0\}$$

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and

$$\mathscr{D}_R^{\geq 0} = R^{\perp_{<0}} := \{X \in \mathsf{D^b}(R) \mid \mathsf{Hom}_{\mathsf{D^b}(R)}(R, X[i]) = 0 \ \forall i < 0\}$$

It is easy to check that the heart $\mathcal{H}_R$ is equivalent to $\mathsf{Mod}\text{-}R$. Moreover, if E is an injective cogenerator of $\mathsf{Mod}\text{-}R$ then we also have $\mathscr{D}_R^{\leq 0} = {}^{\perp_{<0}}E$ and $\mathscr{D}_R^{\geq 0} = {}^{\perp_{>0}}E$.

Consider now a derived equivalence $\phi \colon \mathsf{D}^b(R) \xrightarrow{\simeq} \mathsf{D}^b(S)$. Then the standard t-structure is mapped to

$$(\mathscr{D}_R^{\leq 0}, \mathscr{D}_R^{\geq 0}) \stackrel{\phi}{\longmapsto} (\phi(R)^{\perp_{>0}}, \phi(R)^{\perp_{<0}}) = (T^{\perp_{>0}}, T^{\perp_{<0}}) = ({}^{\perp_{<0}}\phi(E), {}^{\perp_{>0}}\phi(E))$$

and the associated hearts are equivalent, i.e. $\mathsf{Mod}\text{-}R$ is equivalent with $T^{\perp_{>0}} \cap T^{\perp_{<0}} \simeq \mathsf{Mod}\text{-}\mathsf{End}_{\mathsf{D}^b(S)}(T)$ since T is a small projective generator in $T^{\perp_{>0}} \cap T^{\perp_{<0}}$.

Our goal in this note is to generalize derived equivalences from rings to abelian categories using the approach of t-structures. But more precisely:

Aim. Obtain derived equivalences between abelian categories from not necessarily compact tilting and cotilting objects.

We highlight the cotilting case for getting derived equivalences between Grothendieck categories, see Corollary 3.3 (ii), and also that these objects can be really "big". We refer also to [7] for derived equivalences induced by big cotilting modules.

2. (Co)Silting Objects in Triangulated Categories

Let $\mathcal{T}$ be a triangulated category. For an object X in $\mathcal{T}$, we denote by $\mathsf{Add}(X)$ (resp. $\mathsf{Prod}(X)$) the full subcategory of $\mathcal{T}$ consisting of all objects which are summands of a direct sum (resp. of a direct product) of X. From now on we assume that $\mathcal{T}$ has set-indexed coproducts and set-indexed products.

The main notion of this note is the following.

Definition 2.1. An object X in $\mathcal{T}$ is called:

(i) **silting** if $(X^{\perp_{>0}}, X^{\perp_{<0}})$ is a t-structure in $\mathcal{T}$ and $X \in X^{\perp_{>0}}$.
(ii) **cosilting** if $({}^{\perp_{<0}}X, {}^{\perp_{>0}}X)$ is a t-structure in $\mathcal{T}$ and $X \in {}^{\perp_{>0}}X$.
(iii) **tilting** if it is silting and $\mathsf{Add}(X) \in X^{\perp_{\neq 0}} = X^{\perp_{>0}} \cap X^{\perp_{<0}}$.
(iv) **cotilting** if it is cosilting and $\mathsf{Prod}(X) \in {}^{\perp_{\neq 0}}X = {}^{\perp_{<0}}X \cap {}^{\perp_{>0}}X$.

We say that a t-structure is **silting**, **cosilting**, **tilting** or **cotilting** if and only if it arises from a silting, cosilting, tilting or a cotilting object, respectively.

The above definition is mainly inspired by the works of Aihara-Iyama [1], Angeleri Hügel-Marks-Vitória [2] and Wei [8]. It should be noted that there is no "compactness" in the above defition. For more explanations on the above definition see [6]. We provide now some examples.

Example 2.2. (i) Let $\mathscr{A}$ be an abelian category with a projective generator P. Then P is a tilting object in $\mathsf{D^b}(\mathscr{A})$. Indeed, since for a complex X in $\mathsf{D^b}(\mathscr{A})$ we have $H^i(X) = 0$ if and only if $\mathsf{Hom}_{\mathsf{D^b}(\mathscr{A})}(P, X[i]) = 0$ (see [6]), it follows that $(P^{\perp_{>0}}, P^{\perp_{<0}})$ is a t-structure and of course P lies in the heart $\mathcal{H}_P$. Dually, an injective cogenerator in an abelian category $\mathscr{A}$ is a cotilting object in $\mathsf{D^b}(\mathscr{A})$.
(ii) For a ring R, any (large) tilting, resp. cotilting, module in $\mathsf{Mod}\text{-}R$ is tilting, resp. cotilting, in $\mathsf{D}(\mathsf{Mod}\text{-}R)$. We refer to [2], resp. [7], for more details.
(iii) Any silting complex in the sense of Wei [8] in $\mathsf{D}(\mathsf{Mod}\text{-}R)$ is silting according to Definition 2.1, see [2].
(iv) Let $\mathcal{T}$ be a triangulated category with coproducts. If $\mathcal{M} = \mathsf{add}\, M$ is a silting subcategory of $\mathcal{T}$ in the sense of Aihara-Iyama [1], that is, M is a compact silting object, then from [1, Theorem 4.10] the object M is silting according to Definition 2.1.

Remark 2.3. Let X be a silting object in $\mathcal{T}$.

(i) The object X is a generator in $\mathcal{T}$, that is, $X^{\perp_{\mathbb{Z}}} = \{Y \in \mathcal{T} \mid \mathsf{Hom}_{\mathcal{T}}(X, Y[k]) = 0, \forall k \in \mathbb{Z}\} = 0$.
We sketch the proof. Let Y be an object in $X^{\perp_{\mathbb{Z}}}$. We show that $Y = 0$. From the t-structure $(X^{\perp_{>0}}, X^{\perp_{<0}})$ there is a triangle:

$$\tau_X^{\leq -1}(Y) \longrightarrow Y \longrightarrow \tau_X^{\geq 0}(Y) \longrightarrow \tau_X^{\leq -1}(X)[1] \tag{2.1}$$

where $\tau_X^{\leq -1}(Y) \in X^{\perp_{>0}}[1]$ and $\tau_X^{\geq 0}(Y) \in X^{\perp_{<0}}$. Applying the cohomological functor $\mathsf{Hom}_{\mathcal{T}}(X,-)$ (we just write $(X,-)$) to (2.1) we obtain the following long exact sequence:

$$\cdots \longrightarrow (X,Y) \longrightarrow (X,\tau_X^{\geq 0}(Y)) \longrightarrow (X,\tau_X^{\leq -1}(X)[1]) \longrightarrow (X,Y[1]) \longrightarrow (X,\tau_X^{\geq 0}(Y)[1]) \longrightarrow \cdots$$

Since $(X,Y[1]) = 0$ and $(X,\tau_X^{\leq -1}(X)[2]) = 0$ it follows that $(X,\tau_X^{\geq 0}(Y)[1]) = 0$. Continuing in this way, we get that $\tau_X^{\geq 0}(Y) \in X^{\perp_{>0}}$. But since $\tau_X^{\geq 0}(Y) \in X^{\perp_{<0}}$, we infer that $\tau_X^{\geq 0}(Y) = 0$. Similarly, we derive that $\tau_X^{\leq -1}(Y) = 0$ and therefore from the triangle (2.1) we have $Y = 0$.

(ii) The t-structure $(X^{\perp_{>0}}, X^{\perp_{<0}})$ is non-degenerate and therefore it can be described as follows:

$$X^{\perp_{>0}} = \{Y \in \mathcal{T} \mid H_X^k(Y) = 0, \forall k > 0\} \quad \text{and} \quad X^{\perp_{<0}} = \{Y \in \mathcal{T} \mid H_X^k(Y) = 0, \forall k < 0\}$$

In particular, the fact that the object X is a generator in $\mathcal{T}$ implies that if an object Y lies in $X^{\perp_{>0}}[k]$ for all $k \in \mathbb{Z}$, then Y is zero. Similarly, $Y = 0$ if it lies in $X^{\perp_{<0}}[k]$ for all $k \in \mathbb{Z}$.

We continue with the next result which provides a characterisation of silting objects in derived categories of Grothendieck categories. For the proof see [6].

Proposition 2.4. *Let X be an object in $\mathsf{D}(\mathscr{A})$ where $\mathscr{A}$ is a Grothendieck category. Then X is silting if and only if the following conditions hold:*

(i) $\mathsf{Hom}_{\mathsf{D}(\mathscr{A})}(X, X[k]) = 0$ *for all* $k > 0$.
(ii) X *generates* $\mathsf{D}(\mathscr{A})$.
(iii) $X^{\perp_{>0}}$ *is closed for coproducts.*

An interesting aspect of the notion of silting and cosilting (Definition 2.1) is verified also by the following result. More precisely, we show that these objects produce hearts with very nice properties.

Proposition 2.5. *Let $\mathcal{T}$ be a triangulated category.*

(i) *Let X be a silting object in $\mathcal{T}$. Then the object $H_X^0(X)$ is a projective generator in $\mathcal{H}_X = X^{\perp_{\neq 0}}$.*
(ii) *Let X be a cosilting object in $\mathcal{T}$. Then the object $H_X^0(X)$ is an injective cogenerator in $\mathcal{H}_X = {}^{\perp_{\neq 0}}X$.*

Proof. We sketch the proof of (i), part (ii) follows dually. We refer to [6] for more details.

• $H_X^0(X)$ is projective in $\mathcal{H}_X$: It suffices to show that $\mathsf{Ext}^1_{\mathcal{H}_X}(H_X^0(X), Y) = 0$ for all $Y \in \mathcal{H}_X$. From [4] this is equivalent to $\mathsf{Hom}_{\mathcal{T}}(H_X^0(X), Y[1]) = 0$. For the silting object X consider the truncation triangle

$$\tau_X^{\leq -1}(X) \longrightarrow X \longrightarrow \tau_X^{\geq 0}(X) \longrightarrow \tau_X^{\leq -1}(X)[1] \tag{2.2}$$

and note that $H_X^0(X) = \tau_X^{\geq 0}(X)$. Applying the functor $\mathsf{Hom}_{\mathcal{T}}(-, Y[1])$ to the triangle (2.2), it follows that $\mathsf{Hom}_{\mathcal{T}}(H_X^0(X), Y[1]) = 0$.

• $H_X^0(X)$ is generator in $\mathcal{H}_X$: Let Y be an object in $\mathcal{H}_X$. If we apply the cohomological functor $\mathsf{Hom}_{\mathcal{T}}(-, Y)$ to the triangle (2.2), we obtain the isomorphism $\mathsf{Hom}_{\mathcal{T}}(H_X^0(X), Y) \cong \mathsf{Hom}_{\mathcal{T}}(X, Y)$. Since Y lies in the heart $\mathcal{H}_X$ and X is a generator in $\mathcal{T}$, see Remark 2.3 (i), we get that $\mathsf{Hom}_{\mathcal{T}}(X, Y) \neq 0$. □

3. The Realisation Functor and Derived Equivalences

Let $\mathcal{T}$ be a triangulated category with a t-structure $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ and $\mathcal{H} = \mathcal{T}^{\leq 0} \cap \mathcal{T}^{\geq 0}$ its heart. We want a t-exact functor $\mathsf{D}^{\mathsf{b}}(\mathcal{H}) \longrightarrow \mathcal{T}$, with respect to the standard t-structure in $\mathsf{D}^{\mathsf{b}}(\mathcal{H})$ and the fixed one in $\mathcal{T}$, that restricts to the identity functor on $\mathcal{H}$. Such a functor was constructed by Beilinson-Bernstein-Deligne in [4] and it is called the **realisation functor** $\mathsf{real}\colon \mathsf{D}^{\mathsf{b}}(\mathcal{H}) \longrightarrow \mathcal{T}$. Their construction depends on the filtered derived category and moreover it is assumed that $\mathcal{T}$ is a full triangulated subcategory of the derived category of an abelian category. It should be noted that there is a more general approach due to Beilinson [3] via the notion of f-categories. We refer to [3], as well as to [6], for more details on f-categories. We start by recalling very briefly the construction of filtered derived categories.

For an abelian category $\mathscr{A}$, we consider the category $\mathsf{CF}(\mathscr{A})$ of complexes in $\mathscr{A}$ with a finite decreasing filtration. Thus, the objects of $\mathsf{CF}(\mathscr{A})$ are complexes $X \in \mathsf{C}(\mathscr{A})$ such that there is a filtration F of X which looks as follows:

$$X = F_k X \supseteq \cdots \supseteq F_{-n}X \ \supseteq \ F_{-n+1}X \ \supseteq \ \cdots \ \supseteq \ F_{m-1}X \ \supset \ F_m X = 0$$

The morphisms between two objects of $\mathsf{CF}(\mathscr{A})$ are morphisms of complexes which respect the filtrations of the objects. Note that $\mathsf{CF}(\mathscr{A})$ is an additive category with kernels and cokernels but it is not abelian.

Let X and Y be two object in $\mathsf{CF}(\mathscr{A})$. A morphism $f\colon X \longrightarrow Y$ is a quasi-isomorphism if the induced morphism $F_n f\colon F_n X \longrightarrow F_n Y$ is a quasi-isomorphism for all integers n. Then the localisation of $\mathsf{CF}(\mathscr{A})$ with respect to all quasi-isomorphisms is the **filtered derived category** $\mathsf{DF}(\mathscr{A})$. Note that $\mathsf{DF}(\mathscr{A})$ inherits a structure of a triangulated category. We refer to [4] and [5] for more details on filtered derived categories.

Let $\mathscr{A}$ be an abelian category and $\mathsf{D^b}(\mathscr{A})$ its bounded derived category. Assume that $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ is a t-structure in $\mathsf{D^b}(\mathscr{A})$ with heart $\mathcal{H} = \mathcal{T}^{\leq 0} \cap \mathcal{T}^{\geq 0}$. Beilinson-Bernstein-Deligne have proved that there is a unique t-structure in the filtered derived category $\mathsf{DF^b}(\mathscr{A})$ whose heart, denoted by $\mathsf{HF^b}(\mathscr{A})$, is equivalent to the category $\mathsf{C^b}(\mathcal{H})$ of bounded complexes over $\mathcal{H}$. We refer to [3,4] for details on this equivalence. Then, they showed that the vetical composition of functors in the following diagram sends quasi-isomorphisms to isomorphisms, and therefore there is a unique exact functor, called the **realization functor** and denoted by real, which makes the diagram commutative:

$$\begin{tikzcd}
\mathsf{C^b}(\mathcal{H}) \arrow[r, "q"] & \mathsf{D^b}(\mathcal{H}) \arrow[ddd, bend left, "\mathsf{real}"] \\
\mathsf{HF^b}(\mathscr{A}) \arrow[u, "\simeq"] \arrow[d, hook] & \\
\mathsf{DF^b}(\mathscr{A}) \arrow[d, "\text{forgetful}"'] & \\
\mathsf{D^b}(\mathscr{A}) &
\end{tikzcd}$$

We summarize in the next result properties of the realization functor.

Theorem 3.1. *[4, Section 3.1] Let $\mathscr{A}$ be an abelian category and $\mathsf{D^b}(\mathscr{A})$ its bounded derived category. Let $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ be a t-structure in $\mathsf{D^b}(\mathscr{A})$ with heart $\mathcal{H} = \mathcal{T}^{\leq 0} \cap \mathcal{T}^{\geq 0}$. Then there is a triangle functor $\mathsf{real}\colon \mathsf{D^b}(\mathcal{H}) \longrightarrow \mathsf{D^b}(\mathscr{A})$ such that*

- $\mathsf{real}|_{\mathcal{H}} = \mathrm{Id}_{\mathcal{H}}$, *and*
- $\mathsf{Hom}_{\mathsf{D^b}(\mathcal{H})}(X, Y[n]) \cong \mathsf{Hom}_{\mathsf{D^b}(\mathscr{A})}(X, Y[n])$ *for $n \leq 1$ and for all $X, Y \in \mathcal{H}$.*

Moreover, the following statements are equivalent:

(i) *The functor* real *is a triangle equivalence.*
(ii) *The functor* real *is fully faithful.*
(iii) *We have isomorphisms: $\mathsf{Hom}_{\mathsf{D^b}(\mathcal{H})}(X, Y[n]) \cong \mathsf{Hom}_{\mathsf{D^b}(\mathscr{A})}(X, Y[n])$ for all $n \geq 2$ and $X, Y \in \mathcal{H}$.*
(iv) *Given objects X and Y in $\mathcal{H}$, $n \geq 2$ and a morphism $f\colon X \longrightarrow Y[n]$ in $\mathsf{D^b}(\mathscr{A})$, there is an epimorphism $g\colon Z \longrightarrow X$ in $\mathcal{H}$ for some object $Z \in \mathcal{H}$, such that $g \circ f = 0$ in $\mathsf{D^b}(\mathscr{A})$.*
(v) *Given objects X and Y in $\mathcal{H}$, $n \geq 2$ and a morphism $f\colon X \longrightarrow Y[n]$ in $\mathsf{D^b}(\mathscr{A})$, there is a monomorphism $g\colon Y \longrightarrow Z$ in $\mathcal{H}$ for some object $Z \in \mathcal{H}$, such that $f \circ g[n] = 0$ in $\mathsf{D^b}(\mathscr{A})$.*

The main result of this note is the following. We refer to [6] for the proof as well as for further discussions related to this result. In [6] we define bounded (co)silting objects in the following way. For an abelian category $\mathscr{A}$, a silting object X in $\mathsf{D}(\mathscr{A})$ is called **bounded** if $(X^{\perp_{>0}} \cap \mathsf{D^b}(\mathscr{A}), X^{\perp_{<0}} \cap \mathsf{D^b}(\mathscr{A}))$ is a t-structure in $\mathsf{D^b}(\mathscr{A})$ and the heart $\mathcal{H}_X$ belongs to $\mathsf{D^b}(\mathscr{A})$. Dually we have the notion of bounded cosilting objects.

Theorem 3.2. *(P., Vitoria [6]) Let $\mathscr{A}$ be an abelian category such that $\mathsf{D}(\mathscr{A})$ has coproducts (resp. products). Let X be a bounded silting (resp. cosilting) object in $\mathsf{D^b}(\mathscr{A})$. Then the realization functor:*

$$\mathsf{D^b}(\mathcal{H}) \xrightarrow{\mathsf{real}_X} \mathsf{D^b}(\mathscr{A})$$

is a triangle equivalence if and only if the object X is tilting (resp. cotilting).

As a consequence we have the next result which provides necessary and sufficient conditions for two abelian categories to be derived equivalent, when one of them has either a projective generator or an injective cogenerator.

Corollary 3.3. *Let $\mathscr{A}$ and $\mathscr{B}$ be two abelian categories.*

(i) *Assume that* $\mathscr{A}$ *has a projective generator and that* $\mathsf{D}(\mathscr{A})$ *has coproducts. Then there is a triangle equivalence between* $\mathsf{D}^{\mathsf{b}}(\mathscr{A})$ *and* $\mathsf{D}^{\mathsf{b}}(\mathscr{B})$ *if and only if there is a bounded tilting object* T *in* $\mathsf{D}^{\mathsf{b}}(\mathscr{B})$ *such that the heart* $\mathcal{H}_T$ *is equivalent with* $\mathscr{A}$*.*

(ii) *Assume that* $\mathscr{B}$ *has an injective cogenerator and that* $\mathsf{D}(\mathscr{B})$ *has products. Then there is a triangle equivalence between* $\mathsf{D}^{\mathsf{b}}(\mathscr{A})$ *and* $\mathsf{D}^{\mathsf{b}}(\mathscr{B})$ *if and only if there is a bounded cotilting object* T *in* $\mathsf{D}^{\mathsf{b}}(\mathscr{A})$ *such that the hear* $\mathcal{H}_T$ *is equivalent with* $\mathscr{B}$*.*

Proof. (i) Assume that there is a triangle equivalence $\phi\colon \mathsf{D}^{\mathsf{b}}(\mathscr{A}) \xrightarrow{\simeq} \mathsf{D}^{\mathsf{b}}(\mathscr{B})$ and let P be a projective generator of $\mathscr{A}$. Then, from Example 2.2 (i), $T := \phi(P)$ is a bounded tilting object in $\mathsf{D}^{\mathsf{b}}(\mathscr{B})$ such that the associated heart $\mathcal{H}_T$ is equivalent with $\mathscr{A}$.

Conversely, by Theorem 3.2 there is a triangle equivalence $\mathsf{real}\colon \mathsf{D}^{\mathsf{b}}(\mathcal{H}_T) \xrightarrow{\simeq} \mathsf{D}^{\mathsf{b}}(\mathscr{B})$ and also we have an equivalence $\mathsf{D}^{\mathsf{b}}(\mathcal{H}_T) \simeq \mathsf{D}^{\mathsf{b}}(\mathscr{A})$. We infer that $\mathsf{D}^{\mathsf{b}}(\mathscr{A})$ is triangle equivalent with $\mathsf{D}^{\mathsf{b}}(\mathscr{B})$.

(ii) This follows dually as part (i). □

References

[1] T. Aihara, O. Iyama, *Silting mutation in triangulated categories*, J. Lond. Math. Soc. **85** (2012), no. 3, 633-668.

[2] L. Angeleri Hügel, F. Marks and J. Vitória, *Silting modules*, arXiv:1405.2531.

[3] A. Beilinson, *On the derived category of perverse sheaves*, in *K-theory, arithmetic and geometry (Moscow, 19841986)*, Lecture Notes in Math., 1289, Springer, Berlin (1987).

[4] A. Beilinson, J. Bernstein and P. Deligne, *Faisceaux Pervers*, (French) [Perverse sheaves], Analysis and topology on singular spaces, I (Luminy, 1981), 5–171, Asterisque, **100**, Soc. Math. France, Paris, (1982).

[5] L. Illusie, *Complexe cotangent et déformations. I.*, Lecture notes in Mathematics, **239**, Springer-Verlag, Berlin, 1971.

[6] C. Psaroudakis and J. Vitória, *Realisations functors in Tilting theory*, preprint (2015).

[7] J. Stovicek, *Derived equivalences induced by big cotilting modules*, Adv. Math. **263** (2014), 45–87.

[8] J. Wei, *Semi-tilting complexes*, Israel J. Math. **194**, no. 2 (2013), 871–893.

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