The preprojective algebra — revisited

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Aim of the talk

My talk is going to review work from the 1980’s and 1990’s on the (graded) preprojective algebra of the path algebra of a finite quiver (w/o oriented cycles). Things will be looked upon from a different perspective, and new results will be integrated.
Three incarnations

The (graded) **preprojective algebra** \( \Pi = \Pi(kQ) \), attached to \( kQ \) for a finite quiver \( Q \) w/o oriented cycles comes in three incarnations as the

1. *\( k \)-path algebra of the double of \( Q \) modulo the mesh-relations* [Gelfand-Panomarev, ’79]

2. **tensor algebra** \( T(\Lambda M_\Lambda) \) of the bimodule
   \[ M = \text{Ext}^1_\Lambda(D\Lambda, \Lambda) = \text{TrD} \Lambda \] [BGL ’87]

3. **orbit algebra** \( \bigoplus_{n=0}^{\infty} \text{Hom}(\Lambda, \text{TrD}^n \Lambda) \) [BGL ’87].

Showing equivalence of the definitions, uses shape of the preprojective components as mesh categories from [Happel ’88], see [Ringel ’98].
Let $Q$ be a finite quiver $Q$ w/o oriented cycles. The quiver of the preprojective algebra $\Pi = \Pi(kQ)$ is obtained from $Q$ by adding to each arrow, say $a$, an arrow $a^*$ in the reverse direction.

For each vertex $v$ we request the mesh relations. For $v = 2$ we have, for instance, $\beta^*\beta + \gamma^*\gamma + \alpha\alpha^* = 0$. Old arrows get degree zero, new arrows get degree one.
Let \( R = \bigoplus_{n=0}^{\infty} R_n \) be a positively graded \( k \)-algebra. The **companion category** \([\mathbb{Z}; R]\) is the \( k \)-category with

- objects \( n \) are in 1-to-1 correspondence with the integers \( n \in \mathbb{Z} \).
- morphism spaces are given as \((m, n) = R_{n-m}\).
- composition is induced by the multiplication of \( R \).

(The **positive companion category** \([\mathbb{Z}_+; R]\) is the full subcategory consisting of objects \( n \) for integers \( n \geq 0 \).)

**Lemma**

The categories \(([\mathbb{Z}; R], \text{Ab})\) and \( \text{Mod}^{\mathbb{Z}}-R \) of additive functors (resp. \( \mathbb{Z} \)-graded modules) are equivalent under \( F \mapsto \bigoplus_{n \in \mathbb{Z}} F(n) \).
Functors on the mesh category

$k$ a field, $Q$ a finite connected quiver w/o oriented cycles. $\Lambda = kQ$. $\Pi = \Pi(\Lambda)$. For (2) and (3) use [Happel '88]

Theorem

1. $\mathbb{Z}$-graded (resp. $\mathbb{Z}_+$-graded) modules over the preprojective algebra $\Pi$ are additive functors on the mesh category $k[\mathbb{Z}Q]$ (resp. $k[\mathbb{Z}_+Q]$).

2. If $Q$ is Dynkin, the additive closure of the mesh category $k[\mathbb{Z}Q]$ is equivalent to the bounded derived category $D^b(\text{mod-}kQ)$.

3. If $Q$ is tame or wild, the positive mesh category $k[\mathbb{Z}_+Q]$ is equivalent to the preprojective component $\mathcal{P} = \mathcal{P}(\Lambda)$ of $\text{mod-}kQ$.

For $Q$ Dynkin, see [Brenner-Butler-King '02] for the self-injectivity of ungraded $\Pi$; further Bobinski-Krause’s ('15) abelianization of a discrete derived category.
Shape of ind-$\Lambda$

$\Lambda = kQ \text{ tame or wild}$. Then the shape of the module category is given by:

$$\text{ind-}\Lambda = \mathcal{P} \lor \mathcal{R} \lor \mathcal{I}.$$  

(Notation indicates: morphisms only from left to right!) Here,

1. $\mathcal{P} = \text{indec. preprojective modules},$
2. $\mathcal{R} = \text{indec. regular modules},$
3. $\mathcal{I} = \text{indec. preinjective modules}.$
Classification of finitely presented functors

\( \Lambda = kQ \ \text{tame or wild} \). A functor \( F : \mathcal{P} \to \text{Ab} \) is called \textbf{finitely presented} if there exists an exact sequence \((P_1, -) \to (P_0, -) \to F \to 0\) with \( P_1, P_2 \) from add-\( \mathcal{P} \).

**Theorem (L-'86)**

The category \( \mathcal{H} = \frac{\text{fp}(\mathcal{P}, \text{Ab})}{\text{fl}(\mathcal{P}, \text{Ab})} \) is an abelian hereditary \( k \)-linear Hom-finite, hence \textbf{Krull-Schmidt}. Its indecomposables are the following

1. \( \text{Hom}(P, -) \) with \( P \in \mathcal{P} \).
2. \( \text{Ext}^1(R, -) \) with \( R \in \mathcal{R} \).
3. \( \text{Ext}^1(I, -) \) with \( I \in \mathcal{I} \).

\( \mathcal{H} \) has \textbf{Serre-duality} and a \textbf{tilting object} \( T \) with \( \text{End}(T) = \Lambda \).

**Note:** \( \mathcal{H} = \frac{\text{mod}^{\mathbb{Z}+}-\Pi}{\text{mod}^{\mathbb{Z}+}_0-\Pi} \)
An instance of tilting

This is an instance of **tilting in abelian categories** [Happel-Reiten-Smalø ’96].
Minamoto’s theorem

Theorem (Minamoto ’08)

Let $R = \frac{k\langle x_1, \ldots, x_n \rangle}{\left(\sum_{i=0}^{n} x_i^2\right)}$, $n \geq 3$, be the non-commutative Beilinson algebra (graded in degree one). Then Serre construction yields an abelian $k$-linear category

$$\text{mod}_{\mathbb{Z}}^{-}R \quad \text{mod}_{\mathbb{Z}}^{-}R$$

with Serre duality that is derived equivalent to the module category $\text{mod-}\Lambda$ over the $n$-Kronecker algebra $\lambda = \circ \xrightarrow{x_1} \circ \xrightarrow{x_n} \circ$.

Proof.

The graded $n$-Beilinson algebra and the graded preprojective algebra $\Pi(\Lambda)$ have isomorphic companion categories. Hence statement follows from the 'preprojective theory'. 

□
A finite connected quiver $Q$ is **extended Dynkin** if and only if its underlying graph admits a positive **additive function** $\lambda$.

**Example**

```
    2
   /|
1 — 2 — 3 — 4 — 3 — 2 — 1
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**Lemma**

For $\Lambda = kQ$ tame, there is a linear form on the Grothendieck group $K_0(\text{mod-}\Lambda)$, called **rank**, that is constant on **AR-orbits** and assigns to each indecomposable projective $P(v)$ the value $\lambda(v)$ for the unique additive function on $Q$.

For $\Lambda$ tame, the category $\text{add-}\mathcal{R}$ of regular $\Lambda$-modules is an **abelian length category**, consisting of a 1-parameter family of tubes, indexed by the projective line over $k$. (Assume for this $k = \overline{k}$).
Theorem (BGL ’87, Braun-Hajarnavis ’94, L ’13)
Assume $\Lambda$ tame. Then $\Pi = \Pi(\Lambda)$ has the following properties

1. $\Pi$ is prime, and noetherian on both sides.
2. $\Pi$ has global dimension two and (graded) Krull dimension two.
3. The center $C = C(\Pi)$ is an affine $k$-algebra of Krull dimension two, and $\Pi$ is module-finite over $C$.
4. The center is a graded simple singularity, hence $C = k[x, y, z]/(f)$.
5. $\Pi$ as a $C$-module is maximal Cohen-Macaulay.

Example
If $Q = \tilde{E}_7$ then $f = z^2 + y^3 + x^3y$, where $(|x|, |y|, |z|; |f|) = (4, 6, 9; 18)$. 
Theorem (L ’13)

The center $C(\Pi)$ is isomorphic to the orbit algebra

$$\bigoplus_{n=0}^{\infty} \text{Hom}(P, \text{TrD}^n P)$$

of any preprojective module $P$ of rank one.
Assume $\Lambda = kQ$, where $Q$ has extended Dynkin type $\tilde{\Delta}$, $\Delta = [a, b, c]$ Dynkin, that is, $1/a + 1/b + 1/c > 1$.

**Theorem (L’86, GL’87, BGL ’88, L’13)**

We have three attached positively graded algebras

1. The preprojective algebra $\Pi = \Pi(\Lambda)$, positively $\mathbb{Z}$-graded.
2. The projective coordinate algebra $S = k[x_1, x_2, x_3]/(x_1^a + x_2^b + x_3^c)$, positively graded by the rank one abelian group $L = \langle \vec{x}_1, \vec{x}_2, \vec{x}_3 | a\vec{x}_1 = b\vec{x}_2 = c\vec{x}_3 \rangle$.
3. The center $C(\Pi)$ of the preprojective algebra.

Serre construction, for any of the three graded algebras, yields the category $\text{coh-}\mathbb{X}$ of coherent sheaves on the weighted projective line $\mathbb{X}(a, b, c)$. 
The prime ideal lattice

Theorem (BGL’87)

Assume $\Lambda = kQ$ tame. Then the lattice of two-sided prime ideals of $\Pi(\Lambda)$ looks as follows:

$$
\begin{array}{cccccc}
& & m_1 & m_2 & \cdots & m_{n-1} & m_n \\
& (p_t)_{t \in \mathbb{P}^1} & & & & & \\
(0) & & & & & & \\
\end{array}
$$

The $m_i$ are in 1-to-1 correspondence with the simple $\Lambda$-modules. The $p_t$ form a one-parameter family, members are in 1-to-1 correspondence with the tubes in the category of regular $\Lambda$-modules.
If $\Lambda$ is wild, then the ring-theoretic properties of $\Pi(\Lambda)$ are as bad as possible

1. very far from being noetherian (no Krull-Gabriel dimension)
2. very far from being commutative (no polynomial identity, small center)
3. infinite Gelfand-Kirillov dimension.

In view of these facts, it is surprising that Serre construction, when applied to $\Pi(\Lambda)$, yields a sensible result $\mathcal{H}$.

By [Happel-Unger '05] the exchange graph of $\mathcal{H}$, which agrees with the exchange graph of the cluster category of $\mathcal{H}$ or $\Lambda$, is connected.