The preprojective algebra — revisited

Helmut Lenzing

Universität Paderborn

Auslander Conference Woodshole 2015

Aim of the talk

My talk is going to review work from the 1980's and 1990's on the (graded) preprojective algebra of the path algebra of a finite quiver (w/o oriented cycles).
Things will be looked upon from a different perspective, and new results will be integrated.

Three incarnations

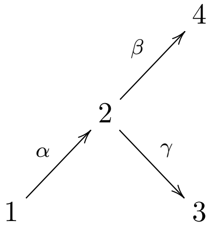
The (graded) **preprojective algebra** $\Pi = \Pi(kQ)$, attached to kQ for a finite quiver Q w/o oriented cycles comes in three incarnations as the

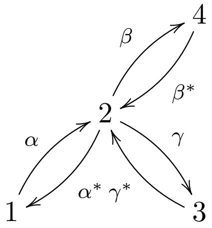
1. k-path algebra of the **double of Q modulo the mesh-relations** [Gelfand-Panomarev, '79]
2. **tensor algebra** $T({}_\Lambda M_\Lambda)$ of the bimodule $M = \mathrm{Ext}^1_\Lambda(D\Lambda, \Lambda) = \mathrm{TrD}\Lambda$ [BGL '87]
3. **orbit algebra** $\bigoplus_{n=0}^{\infty} \mathrm{Hom}(\Lambda, \mathrm{TrD}^n\Lambda)$ [BGL '87].

Showing equivalence of the definitions, uses shape of the preprojective components as mesh categories from [Happel '88], see [Ringel '98].

Preprojective algebra by quiver and relations

Let Q be a finite quiver Q w/o oriented cycles.
The **quiver of the preprojective algebra** $\Pi = \Pi(kQ)$ is obtained from Q by adding to each arrow, say a, an arrow a^* in the reverse direction.





For each vertex v we request the **mesh relations**.
For $v = 2$ we have, for instance, $\beta^*\beta + \gamma^*\gamma + \alpha\alpha^* = 0$.
Old arrows get **degree** zero, new arrows get degree one.

Graded modules are functors

Let $R = \bigoplus_{n=0}^{\infty} R_n$ be a positively graded k-algebra.
The **companion category** $[\mathbb{Z}; R]$ is the k-category with

- objects $\underline{n}$ are in 1-to-1 correspondence with the integers $n \in \mathbb{Z}$.
- morphism spaces are given as $(\underline{m}, \underline{n}) = R_{n-m}$.
- composition is induced by the multiplication of R.

(The **positive companion category** $[\mathbb{Z}_+; R]$ is the full subcategory consisting of objects $\underline{n}$ for integers $n \geq 0$.)

Lemma

The categories $([\mathbb{Z}; R], \mathcal{A}b)$ and $\mathrm{Mod}^{\mathbb{Z}}$-$R$ of additive functors (resp. $\mathbb{Z}$-graded modules) are equivalent under $F \mapsto \bigoplus_{n \in \mathbb{Z}} F(\underline{n})$.

Functors on the mesh category

k a field, Q a finite connected quiver w/o oriented cycles. $\Lambda = kQ$. $\Pi = \Pi(\Lambda)$. For (2) and (3) use [Happel '88]

Theorem

1. $\mathbb{Z}$-graded (resp. $\mathbb{Z}_+$-graded) **modules over the preprojective algebra** Π are **additive functors on the mesh category** $k[\mathbb{Z}Q]$ (resp. $k[\mathbb{Z}_+Q]$).
2. If Q is Dynkin, the additive closure of the **mesh category** $k[\mathbb{Z}Q]$ is equivalent to the bounded derived category $\mathrm{D}^b(\mathrm{mod}\text{-}kQ)$.
3. If Q is tame or wild, the **positive mesh category** $k[\mathbb{Z}_+Q]$ is equivalent to the **preprojective component** $\mathcal{P} = \mathcal{P}(\Lambda)$ of $\mathrm{mod}\text{-}kQ$.

For Q Dynkin, see [Brenner-Butler-King '02] for the self-injectivity of ungraded Π; further Bobinski-Krause's ('15) abelianization of a discrete derived category.

Shape of ind-Λ

$\Lambda = kQ$ **tame or wild**. Then the shape of the module category is given by:

$$\text{ind-}\Lambda = \mathcal{P} \vee \mathfrak{R} \vee \mathcal{I}.$$

(Notation indicates: morphisms only from left to right!) Here,

1. $\mathcal{P}$ = indec. preprojective modules,
2. $\mathcal{R}$ = indec. regular modules,
3. $\mathcal{I}$ = indec. preinjective modules.

Classification of finitely presented functors

$\Lambda = kQ$ **tame or wild**. A functor $F : \mathcal{P} \to \mathcal{A}b$ is called **finitely presented** if there exists an exact sequence $(P_1, -] \to (P_0, -] \to F \to 0$ with P_1, P_2 from add-$\mathcal{P}$.

Theorem (L-'86)

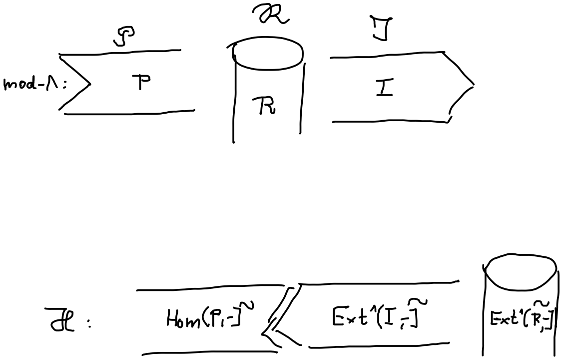
The category $\mathcal{H} = \dfrac{\mathrm{fp}(\mathcal{P}, \mathcal{A}b)}{\mathrm{fl}(\mathcal{P}, \mathcal{A}b)}$ is an abelian hereditary k-linear Hom-finite, hence **Krull-Schmidt**. Its indecomposables are the following

1. $\mathrm{Hom}(P, -]$ with $P \in \mathcal{P}$.
2. $\mathrm{Ext}^1(R, -]$ with $R \in \mathcal{R}$.
3. $\mathrm{Ext}^1(I, -]$ with $I \in \mathcal{I}$.

$\mathcal{H}$ has **Serre-duality** and a **tilting object** T with $\mathrm{End}(T) = \Lambda$.

Note: $\mathcal{H} = \dfrac{\mathrm{mod}^{\mathbb{Z}_+}\text{-}\Pi}{\mathrm{mod}_0^{\mathbb{Z}_+}\text{-}\Pi}$

An instance of tilting



This is an instance of **tilting in abelian categories** [Happel-Reiten-Smalø '96].

Minamoto's theorem

Theorem (Minamoto '08)

Let $R = k\langle x_1, \ldots, x_n\rangle/(\sum_{i=0}^{n} x_i^2)$, $n \geq 3$, be the **non-commutative Beilinson algebra** (graded in degree one). Then Serre construction yields an abelian k-linear category

$$\frac{\mathrm{mod}^{\mathbb{Z}}\text{-}R}{\mathrm{mod}_0^{\mathbb{Z}}\text{-}R}$$

with Serre duality that is derived equivalent to the module category $\mathrm{mod}\text{-}\Lambda$ over the n-**Kronecker algebra** $\lambda = \circ \overset{x_1}{\underset{x_n}{\rightrightarrows}} \circ$.

Proof.

The graded n-Beilinson algebra and the graded **preprojective algebra** $\Pi(\Lambda)$ have isomorphic companion categories. Hence statement follows from the 'preprojective theory'. □

Tame quivers

A finite connected quiver Q is **extended Dynkin** if and only if its underlying graph admits a positive **additive function** λ.

Example

$$\begin{array}{ccccccccccccc} & & & & & & 2 & & & & & & \\ & & & & & & | & & & & & & \\ 1 & — & 2 & — & 3 & — & 4 & — & 3 & — & 2 & — & 1 \end{array}$$

Lemma

For $\Lambda = kQ$ tame, there is a linear form on the Grothendieck group $\mathrm{K}_0(\mathrm{mod}\text{-}\Lambda)$, called **rank**, that is **constant on AR-orbits** and assigns to each indecomposable projective $P(v)$ the value $\lambda(v)$ for the unique additive function on Q.

For Λ tame, the category $\mathrm{add}\text{-}\mathcal{R}$ of regular Λ-modules is an **abelian length category**, consisting of a 1-parameter family of tubes, indexed by the projective line over k. (Assume for this $k = \bar{k}$).

The preprojective algebra — tame case

Theorem (BGL '87, Braun-Hajarnavis '94, L '13)

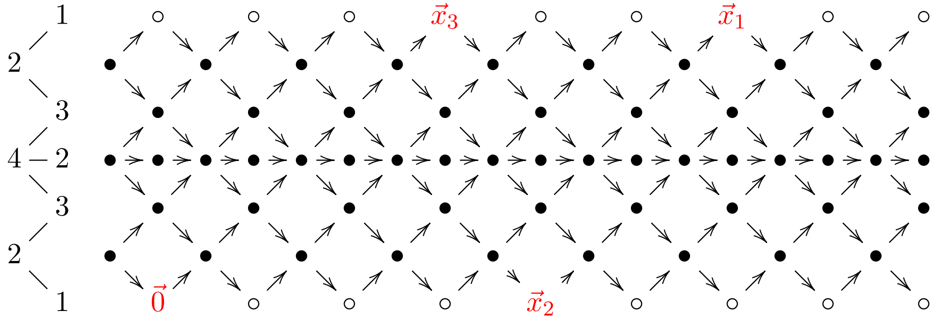
Assume Λ tame. Then $\Pi = \Pi(\Lambda)$ has the following properties

1. Π is **prime**, and **noetherian** on both sides.
2. Π has global dimension two and (graded) Krull dimension two.
3. The center $C = C(\Pi)$ is an affine k-algebra of Krull dimension two, and Π is module-finite over C.
4. The center is a graded simple singularity, hence $C = k[x, y, z]/(f)$.
5. Π as a C-module is maximal Cohen-Macaulay.

Example

If $Q = \tilde{\mathbb{E}}_7$ then $f = z^2 + y^3 + x^3 y$, where $(|x|, |y|, |z|; |f|) = (4, 6, 9; 18)$.

The running example $\tilde{\mathbb{E}}_7$



Theorem (L '13)

The center $C(\Pi)$ is isomorphic to the **orbit algebra** $\bigoplus_{n=0}^{\infty} \mathrm{Hom}(P, \mathrm{TrD}^n P)$ of any preprojective module P of rank one.

Summary for Λ tame

Assume $\Lambda = kQ$, where Q has extended Dynkin type $\tilde{\Delta}$, $\Delta = [a, b, c]$ **Dynkin**, that is, $1/a + 1/b + 1/c > 1$.

Theorem (L'86, GL'87, BGL '88, L'13)

We have three attached positively graded algebras

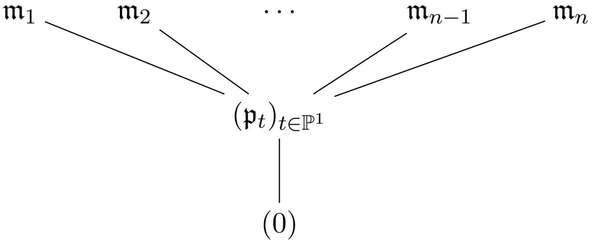
1. The preprojective algebra $\Pi = \Pi(\Lambda)$, positively $\mathbb{Z}$-graded.
2. The projective coordinate algebra $S = k[x_1, x_2, x_3]/(x_1^a + x_2^b + x_3^c)$, positively graded by the rank one abelian group $\mathbb{L} = \langle \vec{x}_1, \vec{x}_2, \vec{x}_3 | a\vec{x}_1 = b\vec{x}_2 = c\vec{x}_3 \rangle$.
3. The center $C(\Pi)$ of the preprojective algebra.

Serre construction, for any of the three graded algebras, yields the category $\mathrm{coh}\text{-}\mathbb{X}$ of **coherent sheaves on the weighted projective line** $\mathbb{X}(a, b, c)$.

The prime ideal lattice

Theorem (BGL'87)

Assume $\Lambda = kQ$ tame. Then the lattice of two-sided prime ideals of $\Pi(\Lambda)$ looks as follows:



The $\mathfrak{m}_i$ are in 1-to-1 correspondence with the simple Λ-modules.
The $\mathfrak{p}_t$ form a one-parameter family, members are in 1-to-1 correspondence with the tubes in the category of regular Λ-modules.

Preprojective algebra for Λ wild

If Λ is wild, then the ring-theoretic properties of $\Pi(\Lambda)$ are as bad as possible

1. very far from being noetherian (no Krull-Gabriel dimension)
2. very far from being commutative (no polynomial identity, small center)
3. infinite Gelfand-Kirillov dimension.

In view of these facts, it is surprising that Serre construction, when applied to $\Pi(\Lambda))$, yields a sensible result $\mathcal{H}$.
By [Happel-Unger '05] the **exchange graph** of $\mathcal{H}$, which agrees with the exchange graph of the cluster category of $\mathcal{H}$ or Λ, is **connected**.