Hopf actions on path algebras

Ryan Kinser [1]   Chelsea Walton [2]

[1]University of Iowa

[2]MIT

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Reference: `arXiv:1410.7696`

Notation: A a $\Bbbk$-algebra, H a Hopf algebra.

Hopf action: Say H acts on A if A is an H-module such that both

- multiplication $A \otimes A \to A$
- inclusion of scalars $\Bbbk \to A$

are H-module morphisms.

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Remark: "Hopf action" can also be defined element-wise, generalizing the definition of "group action" as:

$$g \cdot (ab) = (g \cdot a)(g \cdot b) \quad \text{and} \quad g \cdot 1_A = 1_A$$

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Some types of results:

- Obstructions to existence of actions. For example:
  An action of semi-simple H on commutative domain A must factor through a group algebra if $\operatorname{char} \mathbb{k} = 0$ [Etingof-Walton].

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- Construct examples of actions.
- Parametrize all actions of some class of Hs on class of As.

Today's Hopf algebras: H a Taft algebra $T(n)$ $(n \geq 2)$

Features: Generators g, x.

- Relations $g^n = 1$, $x^n = 0$, $gx = \zeta xg$.
- g must act on A by an order n automorphism.
- x must act on A by nilpotent g-derivation.
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Remark: Our results extend to Frobenius-Lusztig kernel $u_q(\mathfrak{sl}_2)$ and Drinfeld double $D(T(n))$.

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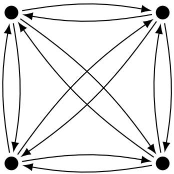
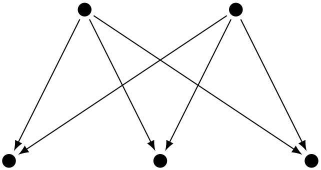
Observation: To admit $T(n)$ action, Q must at least admit $\mathbb{Z}_n$-action.

Examples: Complete quiver K_m and complete bipartite quiver $K_{m,m'}$ for $m, m'|n$.

$K_4 =$ 

$K_{2,3} =$ 

Strategy to classify actions of $H = T(n)$ on $A = \Bbbk Q$:
Identify class of Q which is:

- *Small* enough that we can explicitly describe all H-actions on each $\Bbbk Q$ in the class;
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Definition: Let Q be any quiver with $\mathbb{Z}_n$ action.
A $\mathbb{Z}_n$-component of Q is a maximal $\mathbb{Z}_n$-stable subquiver of Q which is isomorphic to a subquiver of some K_m or $K_{m,m'}$.

Proposition: Every quiver with $\mathbb{Z}_n$-action can be uniquely decomposed into union of $\mathbb{Z}_n$-components.

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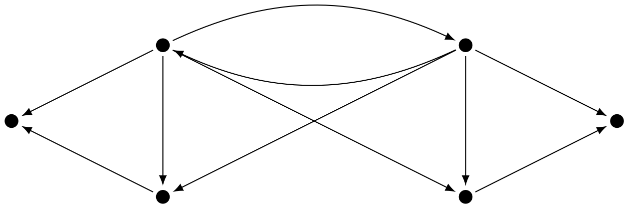
Two more theorems give: Explicit classification of $T(n)$-actions on subquivers of K_m and $K_{m,m'}$.

Roughly: 1 parameter for each arrow, each orbit of vertices, and each orbit of arrows, subject to compatibility conditions. (Please see arXiv:1410.7696 for details.)

The quiver below has $\mathbb{Z}_2$-action by left-right reflection and switching top two arrows.



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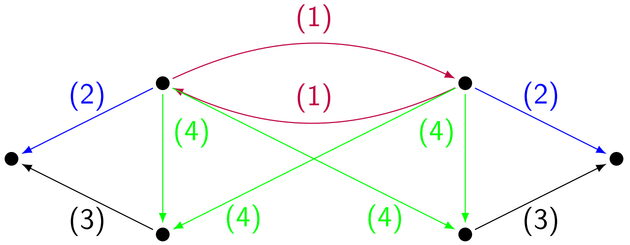

The 4 different colors indicate the $\mathbb{Z}_2$-components.
$T(2)$-actions on this quiver are classified by explicit formulas when restricted to each of these 4 components.

Further results: We also classify all actions of $u_q(\mathfrak{sl}_2)$, and certain actions of $D(T(n))$, on $\Bbbk Q$.

Some future directions:

- Expand class of H (other quantum groups or pointed Hopf algebras)
- Expand class of A (remove restrictions on Q, other finite dimensional algebras)
- Study invariant algebra A^H and smash product algebra $A\#H$