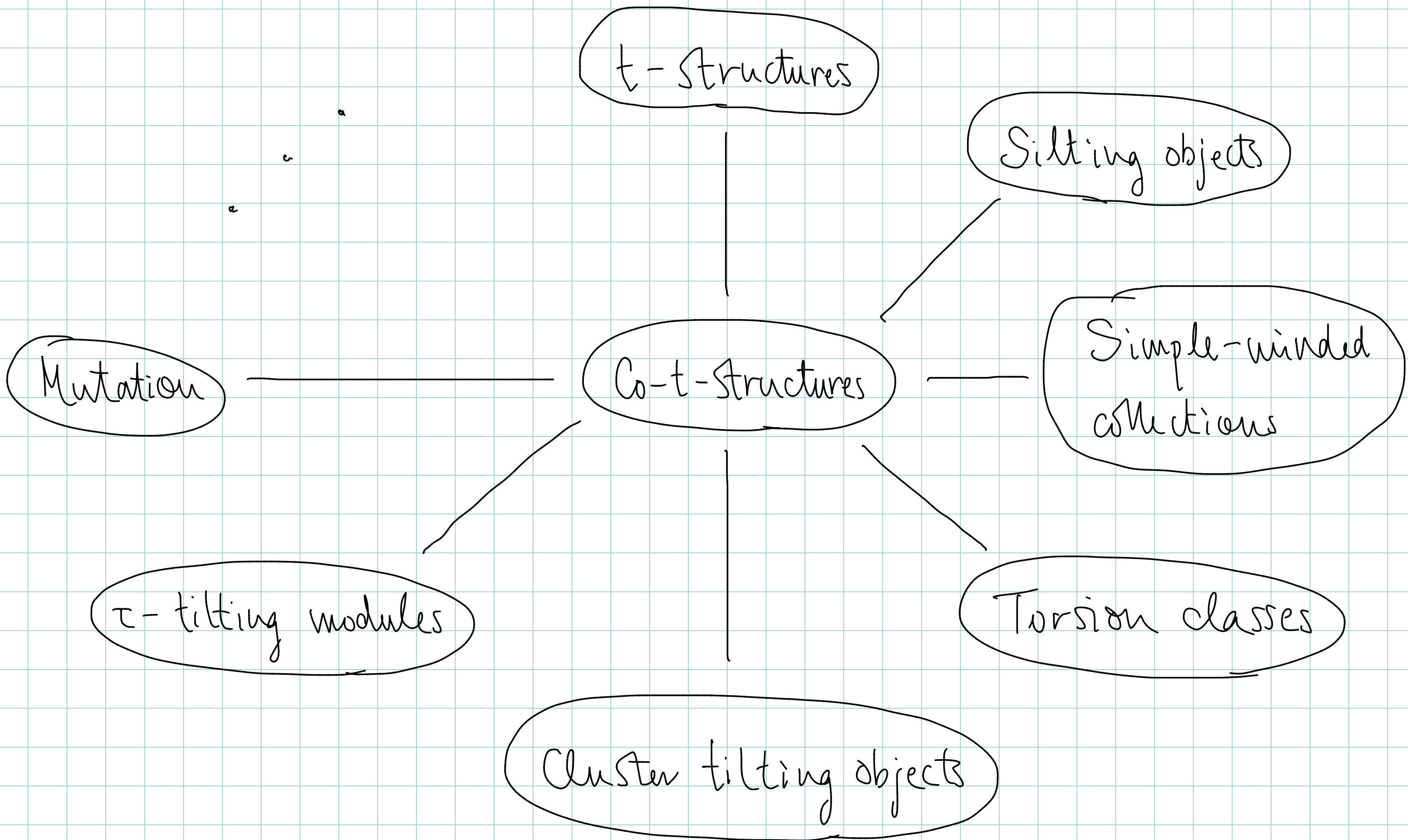
Co-t-structures: The First Decade

0. Overview
1. t-Structures
2. Co-t-Structures
3. Categories skewed towards t- or co-t-Structures
4. The bijections of König and Yang
5. The silting mutation of Aihara and Iyama
6. Intermediate co-t-Structures (after Adachi-Iyama-Reiten)

0. Overview



t-structures
Silting objects
Mutation
Co-t-structures
Simple-minded collections
τ-tilting modules
Torsion classes
Cluster tilting objects

1. t-Structures

- Notion due to Beilinson, Bernstein and Deligne (1982) = [BBD].

- Let R be a ring, $D^b(\mathrm{Mod}\,R)$ the bounded derived category.

- For $x \in D^b(\mathrm{Mod}\,R)$, soft truncation gives a triangle

$$\begin{array}{ccccc} \tau^{\leq 0} d & \longrightarrow & d & \longrightarrow & \tau^{>0} d \\ \cap & & & & \cap \\ D^{\leq 0}(\mathrm{Mod}\,R) = \mathcal{A} & & & & D^{>0}(\mathrm{Mod}\,R) = \mathcal{B} \end{array} \qquad (*)$$

- $(\mathcal{A}, \mathcal{B})$ is a t-structure in $D^b(\mathrm{Mod}\,R)$, that is:

  (1) $\Sigma\mathcal{A} \subseteq \mathcal{A}$, $\Sigma^{-1}\mathcal{B} \subseteq \mathcal{B}$,
  (2) $\mathrm{Hom}(\mathcal{A}, \mathcal{B}) = 0$,
  (3) $\mathcal{A} * \mathcal{B} = D^b(\mathrm{Mod}\,R)$,
  (i.e. each $d \in D^b(\mathrm{Mod}\,R)$ permits a triangle $(*)$)

- <u>Theorem</u> [BBD]. The heart $\mathcal{H} = \mathcal{A} \cap \Sigma\mathcal{B}$ of a t-structure is abelian, with short exact sequences given by the triangles with terms in $\mathcal{H}$. (In the example $\mathcal{H} = \mathrm{Mod}(R)$.)

- <u>Theorem</u> If $(\mathcal{A}, \mathcal{B})$ is a bounded t-structure in $\mathcal{T}$, that is,

$$\mathcal{T} = \bigcup_n \Sigma^n \mathcal{A} = \bigcup_n \Sigma^n \mathcal{B},$$

then the heart $\mathcal{H}$ generates $\mathcal{T}$. In fact,

$$\mathcal{T} = \bigcup_p \Sigma^p \mathcal{H} * \Sigma^{p-1} \mathcal{H} * \cdots * \Sigma^{-p} \mathcal{H}.$$

(Stated in Bridgeland's paper on stability manifolds.)

- <u>Remark</u> It follows from the definition of t-structure that the end terms of the triangle (*) depend functorially on d.

2. Co-t-structures

- Notion independently due to Bondarko and Pauksztello.

- Let R be a ring, $K^b(\operatorname{Prj} R)$ the homotopy category of bounded complexes of projectives.

- For $k \in K^b(\operatorname{Prj} R)$, hard truncation gives a triangle

$$\begin{array}{ccccc} \sigma^{\geq 0}k & \longrightarrow & k & \longrightarrow & \sigma^{<0}k \\ \rotatebox{-90}{$\in$} & & & & \rotatebox{-90}{$\in$} \\ \mathcal{X} & & & & \mathcal{Y} \end{array} \qquad (**)$$

these denote closures under isomorphism in $K^b(\operatorname{Prj} R)$ of $K^{b,\geq 0}(\operatorname{Prj} R)$ and $K^{b,<0}(\operatorname{Prj} R)$

- $(\mathcal{X}, \mathcal{Y})$ is a co-t-structure in $K^b(\operatorname{Prj} R)$, that is,

  (0) $\mathcal{X}$ and $\mathcal{Y}$ are closed under direct summands (automatic for t-structures but not here, and needed for $\mathcal{X}$ and $\mathcal{Y}$ to determine each other),

  (1) $\Sigma^{-1}\mathcal{X} \subseteq \mathcal{X}$, $\Sigma\mathcal{Y} \subseteq \mathcal{Y}$,

  (2) $\operatorname{Hom}(\mathcal{X}, \mathcal{Y}) = 0$,

  (3) $\mathcal{X} * \mathcal{Y} = K^b(\operatorname{Prj} R)$.

  (i.e. each $k \in K^b(\operatorname{Prj} R)$ permits a triangle $(**)$)

- Remark The coheart $\mathcal{C} = \mathcal{X} \cap \Sigma^{-1}\mathcal{Y}$ is not in general abelian, but it does satisfy
① $\mathrm{Hom}(\mathcal{C}, \Sigma^{>0}\mathcal{C}) = 0$. (In the example, $\mathcal{C} = \mathrm{Prj}(R)$.)

- Theorem (Bondarko). If $(\mathcal{X}, \mathcal{Y})$ is a bounded co-t-structure in $\mathcal{T}$, that is,

$$\mathcal{T} = \bigcup_n \Sigma^n \mathcal{X} = \bigcup_n \Sigma^n \mathcal{Y},$$

then the coheart ② $\mathcal{C}$ generates $\mathcal{T}$. In fact,

$$\mathcal{T} = \bigcup_p \Sigma^{-p}\mathcal{C} * \cdots * \Sigma^p \mathcal{C}.$$

- Remark Properties ① and ② imply that $\mathcal{C}$ is a silting subcategory of $\mathcal{T}$, i.e. $\mathrm{Hom}(\mathcal{C}, \Sigma^{>0}\mathcal{C}) = 0$ and thick $\mathcal{C} = \mathcal{T}$. The term "silting" was coined by Keller–Vossieck. Note that tilting subcategories are a special case.

- Remark The end terms of the triangle (**) do not necessarily depend functorially on k.

3. Categories skewed towards t- or co-t-structures

- Philosophy: A triangulated category with a bounded t-structure is "like" a derived category, while one with a bounded co-t-structure is "like" a homotopy category.

- Definition If $\mathcal{T}$ is $\mathbb{C}$-linear then $s \in \mathcal{T}$ is called d-spherical if $\mathcal{T}(s, \Sigma^* s) \cong \mathbb{C}[\varepsilon]/(\varepsilon^2)$ with ε in cohomological degree d.

- Theorem (Keller-Yang-Zhou). Let $d \in \mathbb{Z}$ and consider $A = \mathbb{C}[\varepsilon]/(\varepsilon^2)$ as a DGA with ε in cohomological degree d and zero differential. Then $\mathcal{T} = D^c(A)$ is the unique $\mathbb{C}$-linear Hom-finite algebraic triangulated category which is "thick" of a d-spherical object.

- Theorem (Holm-J-Yang). If $d \geq 1$ then $\mathcal{T}$ has no non-trivial co-t-structures, but one $\mathbb{Z}$-indexed family of non-trivial t-structures. For $d \leq 0$, vice versa.

4. The bijections of König and Yang

– <u>Theorem</u> Let Λ be a finite dimensional $\mathbb{C}$-algebra. There are the following bijections,

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
\boxed{\begin{array}{l}\text{Bounded t-structures}\\ \text{in } D^b(\operatorname{mod}\Lambda) \text{ with length heart}\end{array}} \arrow[r, leftrightarrow] \arrow[d, leftrightarrow, "\text{(a)}", "\begin{array}{l}\text{Al-Nofayee (Student of Rickard),}\\ \text{Buan-Reiten-Thomas,}\\ \text{Keller-Nicolas}\end{array}"'] \arrow[dr, leftrightarrow, "\begin{array}{l}\text{Aihara-Iyama,}\\ \text{Assem-Souto-Trepode, Hoshino-Kato-Miyachi,}\\ \text{Keller-Nicolas, Keller-Vossieck,}\\ \text{Vitoria}\end{array}"] & \boxed{\begin{array}{l}\text{Bounded co-t-structures}\\ \text{in } K^b(\operatorname{prj}\Lambda)\end{array}} \arrow[d, leftrightarrow, "\text{(b)}"', "\begin{array}{l}\text{Aihara-Iyama, Keller-Nicolas,}\\ \text{Mendoza-Saenz-Santiago-Salorio}\end{array}"] \\
\boxed{\begin{array}{l}\text{Isomorphism classes of}\\ \text{simple-minded collections}\\ \text{in } D^b(\operatorname{mod}\Lambda)\end{array}} \arrow[r, leftrightarrow, "\text{Buan-Reiten-Thomas, Rickard}"'] & \boxed{\begin{array}{l}\text{Isomorphism classes of}\\ \text{basic silting obj}^s \text{ in } K^b(\operatorname{prj}\Lambda)\end{array}}
\end{tikzcd}
$$

objects M s.t. $\operatorname{add}(M)$ is a silting subcategory

where

ⓐ $(\mathcal{A},\mathcal{B}) \mapsto$ the simples in $\mathcal{H} = \mathcal{A} \cap \Sigma\mathcal{B}$,

ⓑ $(\mathcal{X},\mathcal{Y}) \mapsto$ a basic additive generator of $\mathcal{C} = \mathcal{X} \cap \Sigma^{-1}\mathcal{Y}$.

5. The silting mutation of Aihara and Iyama

- Let $\mathcal{T}$ be $\mathbb{C}$-linear Hom-finite with split idempotents.

- Let $m = m_0 \oplus m_1$ be a basic silting object with m_0 indecomposable.

- Let $m_0 \xrightarrow{\mu} M_1 \longrightarrow m_0^*$ be a distinguished triangle with μ a minimal $\mathrm{add}(m_1)$-left approximation.

- Theorem m_0^* is indecomposable and the left mutation $m_0^* \oplus m_1$ of m at m_0 is a silting object.

- Theorem There's a symmetric notion of right mutation. Right mutation of $m_0^* \oplus m_1$ takes us back to $m = m_0 \oplus m_1$.

– <u>Definition</u> The silting quiver of $\mathcal{T}$ has a vertex for each isomorphism class of basic silting objects of $\mathcal{T}$ and an arrow $m \longrightarrow m^*$ if m^* is a left mutation of m.

– <u>Example</u> $\mathcal{T} = K^b(\mathrm{prj}\, \mathbb{C}A_2)$ has the following AR quiver.

$$\begin{tikzcd}[column sep=small]
 & X_{-2} \arrow[dr] & & X_0 \arrow[dr] & & X_2 \arrow[dr] & & X_4 \arrow[dr] & & X_6 \arrow[dr] & & \cdots \\
\cdots \arrow[ur] & & X_{-1} \arrow[ur] & & X_1 \arrow[ur] & & X_3 \arrow[ur] & & X_5 \arrow[ur] & & X_7 \arrow[ur] &
\end{tikzcd}$$

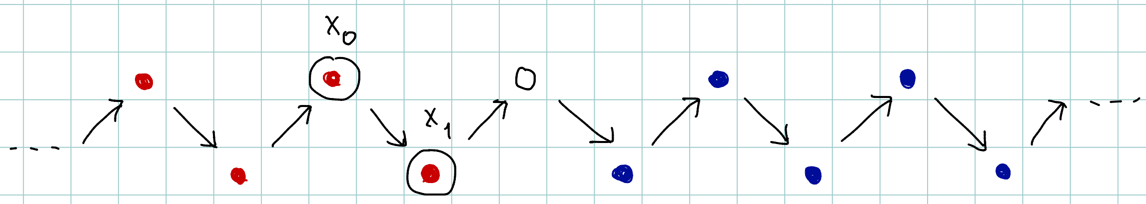
Here is part of the silting quiver, where $m \xrightarrow{\Sigma} n$ means $m \longrightarrow \Sigma n$.

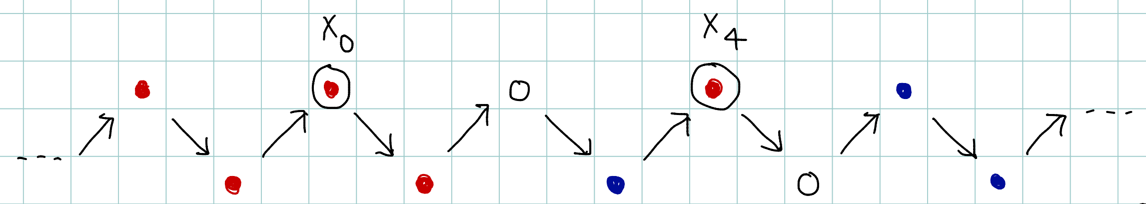
$$\begin{tikzcd}
{} \arrow[r, squiggly] & X_0 \oplus X_1 \arrow[r, bend left, "\text{mut. at } x_1"] & X_0 \oplus X_4 \arrow[l, bend left, "\Sigma", "\text{mut. at } x_0"'] \arrow[r, bend left, "\text{mut. at } x_4"] & X_0 \oplus X_7 \arrow[l, bend left, "\Sigma", "\text{mut. at } x_0"'] & \cdots
\end{tikzcd}$$

Mut. at x_0 gives $x_1 \oplus x_2$

Note: $\Sigma x_i = x_{i+3}$

Here are the co-t-structures corresponding to the first two objects, $\mathcal{X}$ in red, $\mathcal{Y}$ in blue, the coheart circled.





6. Intermediate co-t-structures (after Adachi-Iyama-Reiten)

- Let Λ be a finite dimensional $\mathbb{C}$-algebra.

- Definition Let $t \in \operatorname{mod}\Lambda$ be basic.
  t is called τ-tilting if $\operatorname{Hom}(t, \tau t) = 0$ and
  the number of direct summands in t equals
  $\operatorname{rank} K_0(\operatorname{mod}\Lambda)$.
  t is called support τ-tilting if it is τ-tilting
  over $\Lambda/(e)$ for some idempotent $e \in \Lambda$.

- Definition A co-t-structure $(\mathcal{X}, \mathcal{Y})$ in $K^b(\operatorname{prj}\Lambda)$
  is called intermediate if $\overline{K^{b,\geq 1}(\operatorname{prj}\Lambda)} \subseteq \mathcal{X} \subseteq \overline{K^{b,\geq 0}(\operatorname{prj}\Lambda)}$,
  where overlines denotes closure under isomorphism.

- Definition A silting object m in $K^b(\operatorname{prj}\Lambda)$ is
  called two term if it has the form $P_1 \longrightarrow P_0$.

– <u>Theorem</u> There are the following bijections in $K^b(\mathrm{prj}\,\Lambda)$,

$$\begin{tikzcd}[column sep=small]
\boxed{\begin{array}{c}\text{Intermediate}\\ \text{co-t-structures}\end{array}} \arrow[rr, leftrightarrow, "a"] \arrow[dr, leftrightarrow] & & \boxed{\begin{array}{c}\text{Isomorphism classes of}\\ \text{two term silting objects}\end{array}} \arrow[dl, leftrightarrow, "b"] \\
& \boxed{\begin{array}{c}\text{Isomorphism classes of}\\ \text{support } \tau\text{-tilting modules}\end{array}} &
\end{tikzcd}$$

where

a $(\mathcal{X},\mathcal{Y}) \mapsto$ a basic additive generator of $\mathcal{C} = \mathcal{X} \cap \Sigma^{-1}\mathcal{Y}$,

b $(P_1 \xrightarrow{\partial} P_0) \longmapsto \operatorname{Coker} \partial$.