Maximal Green Sequences via Quiver Semi-Invariants

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Preliminaries

Reference

Joint with Thomas Brüstle, Kiyoshi Igusa and Gordana Todorov.
Preprint on arXiv:1503.07945

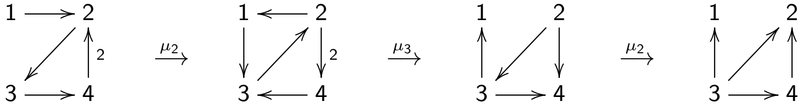
Notation

- K denotes a field,
- Q (acyclic) quiver with
- n vertices.

Mutation

- Fomin and Zelevinsky introduced the notation of *mutation* for (integer) skew-symmetrizable matrices to formalize combinatorial properties of canonical bases / total positivity.
- Can be interpreted as mutation of quivers (skew-symmetric) or more generally valued quivers (skew-symmetrizable).
- Given Q and $k \in \{1, \dots, n\}$ can *mutate* Q in the direction k to form new quiver $\mu_k Q$.

Example (Mutation)



c-Vectors

- Much of the dynamics of cluster mutation encoded in *c-vectors*.
- Form quiver $\widetilde{Q}$ by adding arrows $i \to i'$ (still only mutate at original vertices).
- $\widetilde{Q}' = \mu_k \widetilde{Q}$ contains $\mu_k Q$ as a full subquiver.

Definition

If Q' is obtained from Q by a sequence of mutations, the c-vectors are the vectors $c(i) = (c_1, c_2 \ldots, c_n)$ where

$$c_j(i) = \#(\text{arrows } i \to j' \text{ in } \widetilde{Q}') - \#(\text{arrows } j' \to i \text{ in } \widetilde{Q}')$$

Green Mutation

Theorem (Sign Coherence[1])

Each of the c-vectors either has nonnegative entries or nonpositive entries.

Definition

A vertex $k \in \{1, \ldots, n\}$ of Q' is

- *green* if the k-th c-vector nonnegative, and is
- *red* if the k-th c-vector is nonpositive.

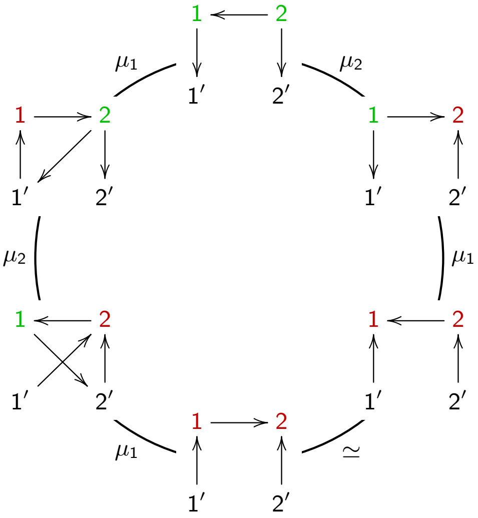
A *maximal green sequence* is a sequence $(k_0, k_1, \ldots, k_m)$ of vertices so that:

1. each k_s is green in $\mu_{k_{s-1}} \circ \cdots \circ \mu_{k_1} \circ \mu_{k_0} Q$, and
2. all indices of $\mu_{k_m} \circ \cdots \circ \mu_{k_1} \circ \mu_{k_0} Q$ are red.

[1] Fomin-Zelevinsky, Derksen-Weyman-Zelevinsky, Igusa-Orr-Todorov-Weyman,Gross-Hacking-Keel-Kontsevich

Green Mutation

Example (Mutation for A_2)



Problems

1. **Does every Q admit a maximal green sequence?**
   - No: e.g., once-punctured torus (Brüstle-Dupont-Pérotin).
   - Yes for Q mutation type A_n (Garver-Musiker).
2. **If Q admits a maximal green sequence, does it only admit finitely many?**
   - Yes for Q tame acyclic (Brüstle-Dupont-Pérotin).
   - Yes for Q mutation equivalent to tame (BHIT).
3. **Can every sequence of green mutations be completed to a maximal green sequence?**
   - In particular, cannot mutate at the source of a multiple arrow (BHIT).
4. **Existence of maximal green sequences preserved under mutation?**
   - No in general: There exist quivers having no maximal green sequences mutation equivalent to quiver which do (Muller).

Results

We will use geometry of quiver semi-invariants to understand maximal green sequences.

Theorem (Source/Target Theorem, Brüstle-H.-Igusa-Todorov)

Let Q be an acyclic valued quiver and $(k_0, k_1, \dots, k_m)$ a maximal green sequence. Then at each step, k_s is not the source of an infinite type arrow of $\mu_{k_s} \circ \cdots \circ \mu_{k_0} Q$.

Theorem (Finiteness Theorem, Brüstle-H.-Igusa-Todorov)

Let Q be a valued quiver mutation equivalent to an acyclic tame valued quiver. Then Q admits only finitely many maximal green sequences.

Semi-Invariant Domains

Definition

1. The *Euler matrix* E has entries

$$E_{ij} = \dim_K \operatorname{Hom}(S_i, S_j) - \dim_K \operatorname{Ext}(S_i, S_j)$$

   where S_i is the simple representation supported at i.

2. Have the *Euler-Ringel* bilinear form

$$\langle \ , \ \rangle : \mathbb{Z}^n \otimes \mathbb{Z}^n \to \mathbb{Z}^n$$

   given by $\langle \alpha, \beta \rangle = \alpha^t E \beta$.

3. For representations M and N have

$$\langle \mathbf{dim} M, \mathbf{dim} N \rangle = \dim_K \operatorname{Hom}(M, N) - \dim_K \operatorname{Ext}(M, N)$$

Semi-Invariant Domains

Definition

A vector $\beta \in \mathbb{Z}^n$ is a *root* of Q if there is an indecomposable β-dimensional representation of Q. A root β is

1. *real* if $\langle \beta, \beta \rangle > 0$
2. *Schur* if $\text{End}(M_\beta) = K$ general M_β with $\mathbf{dim} M_\beta = \beta$.

A root β' is a *subroot* of β if a general β-dimensional representation has a β'-dimensional subrepresentation.

Definition

Let β be a real Schur root. The *semi-invariant domain*

$$D(\beta) = \{x \in \mathbb{R}^n : \langle x, \beta \rangle = 0 \text{ and } \langle x, \beta' \rangle \leq 0 \text{ for all } \beta' \subseteq \beta\}$$

Semi-Invariant Domains

- Consider $Rep(Q, \alpha)$ the space of α-dimensional representations.
- Carries an action of $GL(\alpha)$. Orbits correspond to isomorphism classes of representations.
- A polynomial $f : Rep(Q, \alpha) \to K$ is *semi-invariant* of weight β if $(g \cdot f)(x) = \det(g)^{\beta} f(x)$.

Theorem (Virtual Stability Theorem, Igusa-Orr-Todorov-Weyman[2])

The integer points of the domain $D(\beta)$ are the $\alpha \in \mathbb{Z}^n$ such that $Rep(Q, \alpha)$ has a semi-invariant of weight β.

(Really should use presentation spaces / virtual representation spaces for $\alpha \in \mathbb{Z}^n$.)

[2] cf. also King, Derksen-Weyman

Semi-Invariant Pictures

The domains $D(\beta)$ give a simplicial fan in $\mathbb{R}^n$ with walls labeled by real Schur roots.

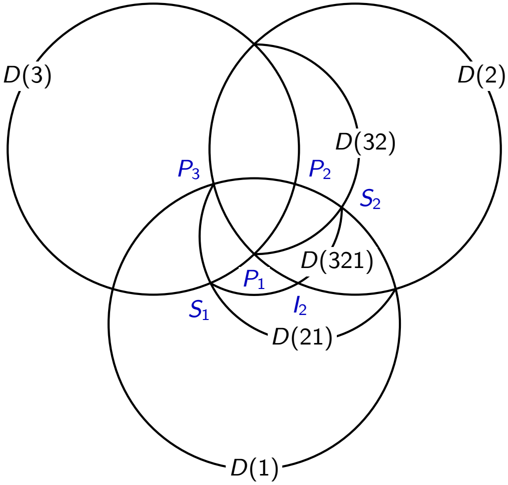
Definition

The *semi-invariant picture* $L(Q)$ is obtained by stereographic projection of $\bigcup_\beta D(\beta) \cap S^{n-1}$.

- Walls in $L(Q)$ have a specified normal orientation given by curvature.
- Internal vertices $L(Q)$ in bijection with rigid modules.
- When Q infinite representation type, $L(Q)$ is a simplicial complex with $\overline{L(Q)} = S^{n-1}$.

Semi-Invariant Pictures

Example (Semi-Invariant Picture for $3 \leftarrow 2 \leftarrow 1$)



Mutation via Semi-Invariant Pictures

Theorem (Igusa-Orr-Todorov-Weyman[3])

Consider the cluster category $\mathcal{C}_Q = \mathcal{D}^b(KQ)/\tau^{-1} \circ [1]$. There are bijections:

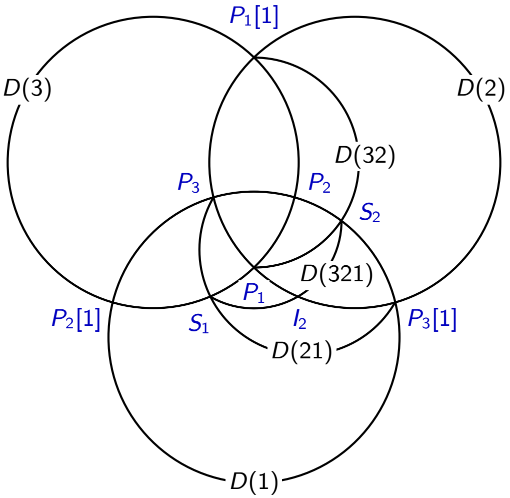
1. $\{$*real Schur roots of* $Q\} \leftrightarrow \{$*walls of* $L(Q)\}$
2. $\{$*indecomposable rigids of* $\mathcal{C}_Q\} \leftrightarrow \{$*vertices of* $L(Q)\}$

So that:

3. $T_1 \oplus \cdots \oplus T_n$ *is a cluster tilting object in* $\mathcal{C}_Q$ *if and only if corresponding vertices span a* $(n-1)$*-simplex in* $L(Q)$
4. *Clusters differ by a mutation if and only if the corresponding simplices share a wall.*

[3] cf. also Schofield, Derksen-Weyman

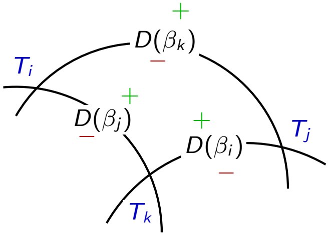
Example (Clusters for $3 \leftarrow 2 \leftarrow 1$)



c-Vector Theorem

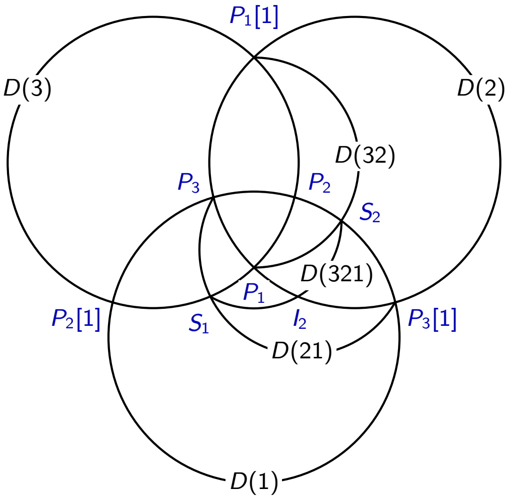
Theorem (c-Vector Theorem, Igusa-Orr-Todorov-Weyman)
Suppose $T = T_1 \oplus \cdots \oplus T_n$ is a cluster and the wall opposite of T_i is $D(\beta_i)$. Then the c-vectors associated to T are $\pm\beta_i$ with sign determined by normal orientation of $D(\beta_i)$.

- A mutation is green if and only if crossing from "outside" of a wall to "inside."
- Maximal green sequence goes from simplex labelled $KQ[1]$ to simplex labelled KQ.



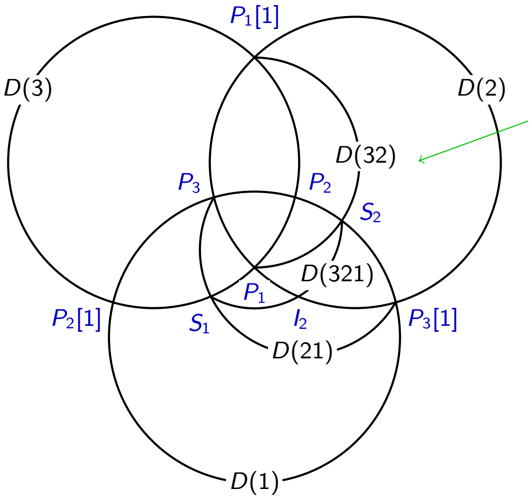
Mutation via Semi-Invariant Pictures

Example ($3 \leftarrow 2 \leftarrow 1$)



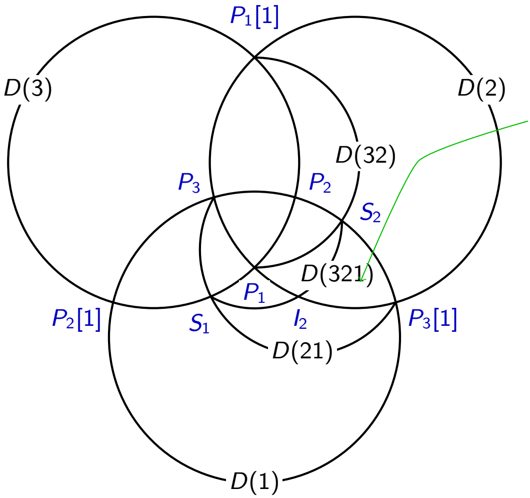
Mutation via Semi-Invariant Pictures

Example $(3 \leftarrow 2 \leftarrow 1)$



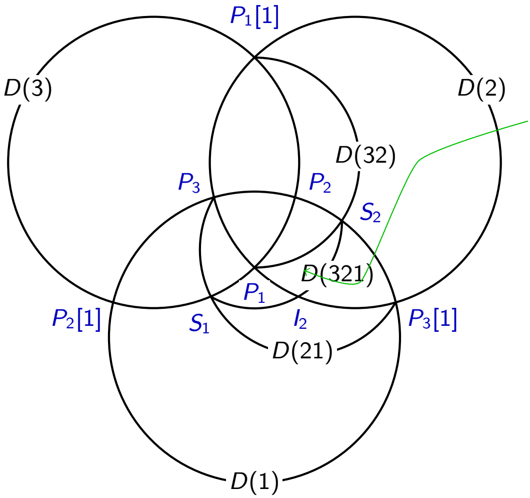
Mutation via Semi-Invariant Pictures

Example $(3 \leftarrow 2 \leftarrow 1)$



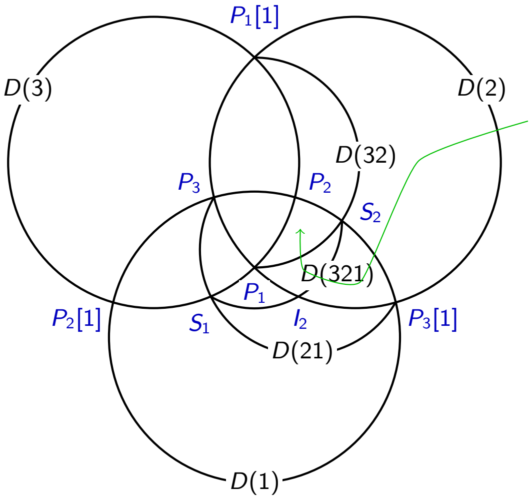
Mutation via Semi-Invariant Pictures

Example $(3 \leftarrow 2 \leftarrow 1)$



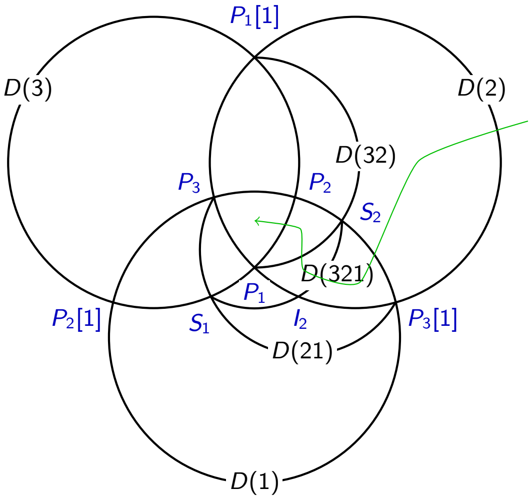
Mutation via Semi-Invariant Pictures

Example $(3 \leftarrow 2 \leftarrow 1)$



Mutation via Semi-Invariant Pictures

Example $(3 \leftarrow 2 \leftarrow 1)$



Proofs of Theorems

Now sketch proofs of the following theorems using semi-invariant pictures.

Theorem (Source/Target Theorem, Brüstle-H.-Igusa-Todorov)

If Q is acyclic $(k_0, k_1, \ldots, k_m)$ *is a maximal green sequence. Then at each step,* k_s *is not the source of an infinite type arrow of* $\mu_{k_s} \circ \cdots \circ \mu_{k_0} Q$.

Theorem (Finiteness Theorem, Brüstle-H.-Igusa-Todorov)

If Q is mutation equivalent to an acyclic tame quiver, then Q admits only finitely many maximal green sequences.

The Rotation Lemma

Both theorems rely on the Rotation Lemma:

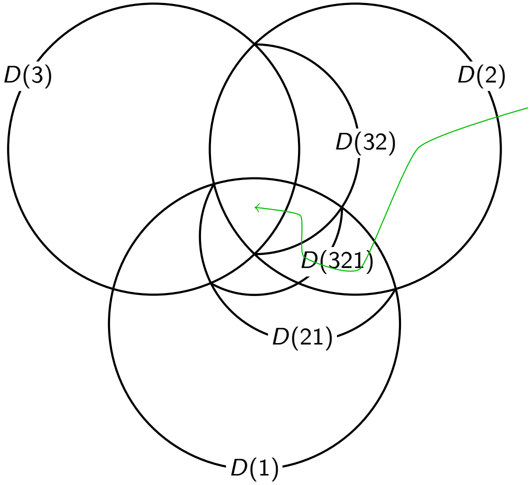
Theorem (Rotation Lemma, BHIT)

If $(k_0, \ldots, k_m)$ *is a maximal green sequence for* Q*, then*

$$(k_1, \ldots, k_m, \sigma^{-1}(k_0))$$

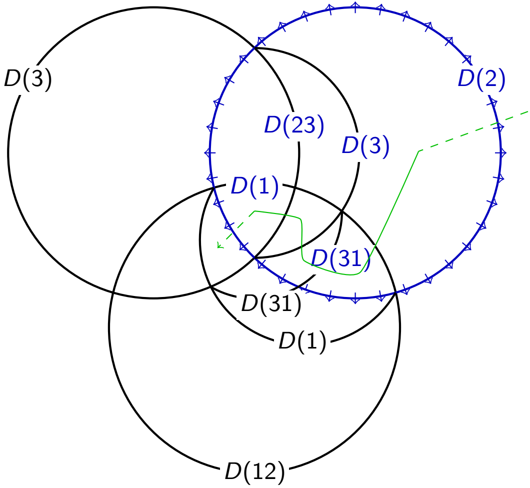
is a maximal green sequence for $\mu_{k_0} Q$ *where* σ *is the permutation associated to the maximal green sequence.*

Example (Original maximal green sequence)



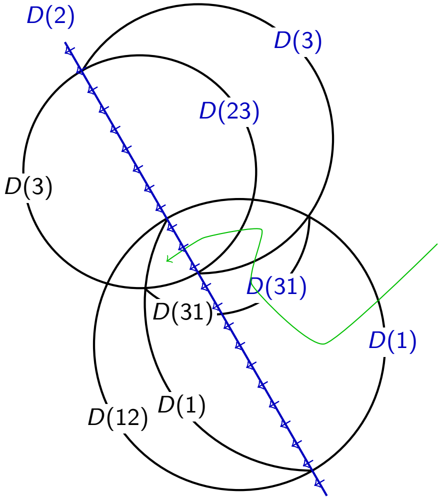
The Rotation Lemma

Example (Rotating the Maximal Green Sequence: Step 1)



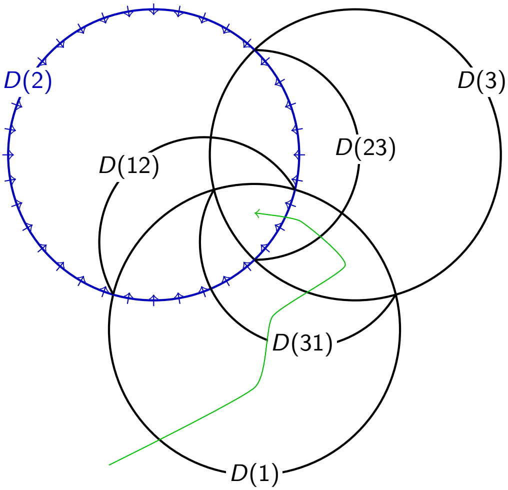
The Rotation Lemma

Example (Rotating the Maximal Green Sequence: Step 2)



The Rotation Lemma

Example (Rotating the Maximal Green Sequence: Step 3)



Proof of Source/Target Theorem

Theorem (Source/Target Theorem)

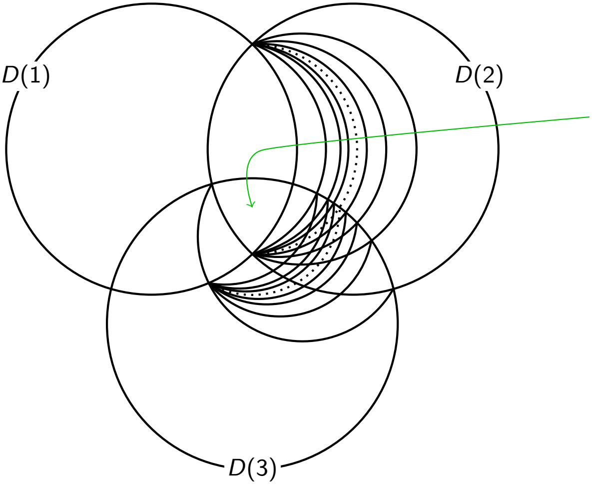
If Q is acyclic $(k_0, k_1, \ldots, k_m)$ is a maximal green sequence. Then at each step, k_s is not the source of an infinite type arrow of $\mu_{k_s} \circ \cdots \circ \mu_{k_0} Q$.

Proof (Sketch)

1. Suppose there is a maximal green sequence where we mutate at the source of a multiple arrow.
2. Use rotation lemma to reduce to case where first mutation is at the source of a multiple arrow, say $2 \rightrightarrows 1$.
3. (Acyclic assumption ensures mutated quiver still has a multiple arrow).
4. In $L(Q)$ have to cross $D(2)$ before $D(1)$.
5. Multiple arrow gives infinite collection of walls to cross.

Proof of Source/Target Theorem

Example (Semi-Invariant Picture for $1 \Leftarrow 2 \leftarrow 3$)



Proof of Finiteness Theorem

Theorem (Finiteness Theorem)

If Q is mutation equivalent to an acyclic tame quiver, then Q admits only finitely many maximal green sequences.

Proof (Sketch)

1. Use rotation lemma to reduce to case of acyclic and tame.
2. Invoke Brüstle-Dupont-Pérotin, or,
3. There is a region in $L(Q)$ that no green sequence can enter/escape, outside of which there are only finitely many walls.

References

[1] T. Brüstle, G. Dupont and M. Pérotin, *On maximal green sequences*.

[2] T. Brüstle, S. Hermes, K. Igusa and G. Todorov, *Semi-invariant pictures and two conjectures about maximal green sequences*.

[3] H. Derksen and J. Weyman, *On the canonical decomposition of quiver representations*.

[4] H. Derksen and J. Weyman, *Semi-invariants for quivers and saturation of Littlewood-Richardson coefficients*.

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[6] S. Fomin and A. Zelevinsky, *Cluster algebras IV: Coefficients*.

[7] M. Gross, P. Hacking, S. Keel and M. Kontsevich, *Canonical bases for cluster algebras*.

[8] K. Igusa, K. Orr, G. Todorov and J. Weyman, *Cluster complexes via semi-invariants*.

[9] K. Igusa, K. Orr, G. Todorov and J. Weyman, *Modulated semi-invariants*.

[10] A. King, *Moduli of representations of finite dimensional algebras*.

[11] G. Muller, *The existence of a maximal green sequence is not invariant under quiver mutation*.

[12] A. Schofield, *Semi-invariants of quivers*.

Thank You!