Combinatorics of Exceptional Sequences in Type $\mathbb{A}$

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(joint with Kiyoshi Igusa, Jacob Matherne, and Jonah Ostroff)

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Outline

Goal: Explicitly describe exceptional sequences of $\Bbbk Q$ where Q is an orientation of

$$1 \text{ —— } 2 \text{ —— } \cdots \text{ —— } n-1 \text{ —— } n.$$

1. Exceptional sequences
2. Strand diagrams
3. Applications

Exceptional sequences

Let $\mathbb{k} = \overline{\mathbb{k}}$ and Q an acyclic quiver.

Definition

An ordered pair of representations (E_1, E_2) of Q is called an **exceptional pair** if

i) each E_i is indecomposable,

ii) $\mathrm{Ext}^1(E_i, E_i) = 0$ for each E_i,

iii) $\mathrm{Hom}(E_2, E_1) = 0, \mathrm{Ext}^1(E_2, E_1) = 0$.

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- A sequence $(E_1, \ldots, E_k)$ $(k \leqslant n := \#Q_0)$ of representations of Q is an **exceptional sequence** if (E_i, E_j) is an exceptional pair for any $i < j$. [Gorodentsev-Rudakov 1987]

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- A set $\{E_1, \ldots, E_k\}$ $(k \leqslant n)$ of representations of Q is an **exceptional collection** if $(E_{\sigma(1)}, \ldots, E_{\sigma(k)})$ is an exceptional sequence for some $\sigma \in \mathfrak{S}_k$.

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- An exceptional sequence or collection is **complete** if $k = n$.

Exceptional sequences

Lemma

The indecomposable representations of a type $\mathbb{A}$ quiver Q are exactly those of the form $X_{i,j}$:

$$\begin{array}{ccccccccccccccccc} 1 & & & & i & & & & & & j & & & & & & n \\ 0 & \overset{0}{—} & \cdots & \overset{0}{—} & 0 & \overset{0}{—} & \mathbb{k} & \overset{1}{—} & \cdots & \overset{1}{—} & \mathbb{k} & \overset{0}{—} & 0 & \overset{0}{—} & \cdots & \overset{0}{—} & 0 \end{array}$$

where $0 \leqslant i < j \leqslant n$.

Example

Consider the quiver $1 \longrightarrow 2 \longleftarrow 3$. The sequence $(X_{0,1}, X_{1,2}, X_{2,3})$ is not a complete exceptional sequence (CES) since

$$\dim_{\mathbb{k}} \mathrm{Ext}^1(X_{2,3}, X_{1,2}) = \#\{3 \xrightarrow{\alpha} 2 \in Q_1\} = 1.$$

The sequence $(X_{2,3}, X_{0,1}, X_{1,2})$ is a CES.

Exceptional sequences

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- If Q is Dynkin, complete exceptional sequences are in bijection with maximal chains in the lattice of noncrossing partitions of W_Q. [Ingalls-Thomas 2009]
- If Q is acyclic, certain types of complete exceptional sequences are in bijection with **c**-matrices of Q. [Speyer-Thomas 2013]

Given a type $\mathbb{A}$ quiver Q,

$$1 \longrightarrow 2 \longleftarrow 3 \longleftarrow 4 \longleftarrow 5$$

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$$1 \xrightarrow{+} 2 \xleftarrow{-} 3 \xleftarrow{-} 4 \xleftarrow{-} 5$$

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Strand diagrams

Fix a type $\mathbb{A}$ quiver and a corresponding ϵ vector. Denote by $\mathcal{S}_{n,\epsilon}$ a collection of $n + 1$ points arranged in a horizontal line.

Can write $\epsilon_i = (x_i, y_i) \in \mathbb{R}^2$.

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$$\underset{\epsilon_0}{\bullet} \qquad \underset{\epsilon_1}{\bullet} \qquad \underset{\epsilon_2}{\bullet} \qquad \cdots \qquad \underset{\epsilon_{n-1}}{\bullet} \qquad \underset{\epsilon_n}{\bullet}$$

Can write $\epsilon_i = (x_i, y_i) \in \mathbb{R}^2$.

Definition

Let $i, j \in [0, n]$ where $i \neq j$. A **strand** $c(i,j)$ on $\mathcal{S}_{n,\epsilon}$ is an isotopy class of simple curves in $\mathbb{R}^2$ where any $\gamma \in c(i,j)$ satisfies:

- the endpoints of γ are ϵ_i and ϵ_j,
- as a subset of $\mathbb{R}^2$, $\gamma \subset \{(x,y) \in \mathbb{R}^2 : x_i \leqslant x \leqslant x_j\} \backslash \{\epsilon_{i+1}, \epsilon_{i+2}, \ldots, \epsilon_{j-1}\}$,
- if $k \in \{0, ..., n\}$ satisfies $i \leqslant k \leqslant j$ and $\epsilon_k = +$ (resp. $\epsilon_k = -$), then γ is locally below (resp. above) ϵ_k.

Strand diagrams

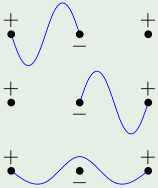
Definition

There is a natural map Φ from $\mathrm{ind}(\mathrm{rep}_{\Bbbk}(Q))$ to the set of strands on $\mathcal{S}_{n,\epsilon}$ given by $\Phi(X_{i,j}) := c(i,j)$.

Example

Let $Q = 1 \longleftarrow 2$.

$X_{0,1}$	$\Bbbk \xleftarrow[-]{0} 0$
$X_{1,2}$	$0 \xleftarrow[-]{0} \Bbbk$
$X_{0,2}$	$\Bbbk \xleftarrow[-]{1} \Bbbk$



Strand diagrams

Let $c(i_1, j_1)$ and $c(i_2, j_2)$ be distinct strands.

Definition

Two strands $c(i_1, j_1)$ and $c(i_2, j_2)$ **intersect nontrivially** if any two curves $\gamma_\ell \in c(i_\ell, j_\ell)$ with $\ell \in \{1, 2\}$ have at least one crossing.

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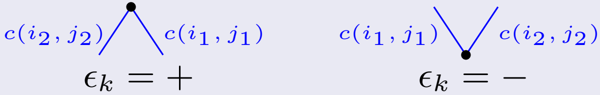
Definition

We say $c(i_2, j_2)$ is **clockwise** from $c(i_1, j_1)$ if and only if any $\gamma_1 \in c(i_1, j_1)$ and $\gamma_2 \in c(i_2, j_2)$ share an endpoint ϵ_k and appear in one of the following two configurations up to isotopy.



Strand diagrams

Definition

A **strand diagram** $d = \{c(i_\ell, j_\ell)\}_{\ell \in [k]}$ $(k \leqslant n)$ on $\mathcal{S}_{n,\epsilon}$ is a collection of strands on $\mathcal{S}_{n,\epsilon}$ that satisfies the following conditions:

- distinct strands do not intersect nontrivially,
- the graph determined by d is a **forest** (i.e. a disjoint union of trees),

Let $\mathcal{D}_{n,\epsilon}$ denote the set of strand diagrams on $\mathcal{S}_{n,\epsilon}$.

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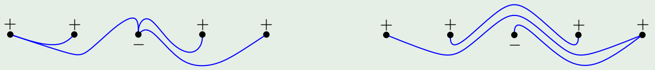
Example

Let $\epsilon = (+,+,-,+,+)$ so that $Q = 1 \xrightarrow{+} 2 \xleftarrow{-} 3 \xrightarrow{+} 4$. Then we have that $d_1 = \{c(0,1), c(0,2), c(2,3), c(2,4)\}$ and $d_2 = \{c(0,4), c(1,3), c(2,4)\}$ are elements of $\mathcal{D}_{4,\epsilon}$.



Main Technical Lemma (G.–Igusa–Matherne–Ostroff)

Let Q and ϵ be given. Fix two distinct indecomposable representations $U, V \in ind(rep_{\mathbb{k}}(Q))$.

1. *The strands $\Phi_{n,\epsilon}(U)$ and $\Phi_{n,\epsilon}(V)$ intersect nontrivially if and only if neither (U, V) nor (V, U) are exceptional pairs.*

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2. *The strand $\Phi_{n,\epsilon}(U)$ is clockwise from $\Phi_{n,\epsilon}(V)$ if and only if (U, V) is an exceptional pair and (V, U) is not an exceptional pair.*
3. *The strands $\Phi_{n,\epsilon}(U)$ and $\Phi_{n,\epsilon}(V)$ do not intersect at any of their endpoints and they do not intersect nontrivially if and only if (U, V) and (V, U) are both exceptional pairs.*

Recall $\mathcal{D}_{n,\epsilon} := \{\text{diagrams } d = \{c(i_\ell, j_\ell)\}_{\ell\in[k]}\}$ and let $\overline{\mathcal{E}}_\epsilon := \{\text{exceptional collections with } k \text{ objects } \overline{\xi}\}$.

Theorem (G.–Igusa–Matherne–Ostroff)

The following map is a bijection

$$\begin{array}{rcl} \overline{\mathcal{E}}_\epsilon & \xrightarrow{\Phi_{n,\epsilon}} & \mathcal{D}_{n,\epsilon} \\ \overline{\xi} = \{X_{i_1,j_1}, \ldots, X_{i_k,j_k}\} & \longmapsto & \{c(i_\ell, j_\ell)\}_{\ell\in[k]}. \end{array}$$

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Example

Let $n = 4$ and $\epsilon = (+,+,-,+,+)$ so that

$$Q = 1 \xrightarrow{+} 2 \xleftarrow{-} 3 \xrightarrow{+} 4$$

$\{X_{0,1}, X_{0,2}, X_{2,3}, X_{2,4}\} \mapsto$ 

Labeled strand diagrams

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A labeled diagram $d(k)$ has a **good labeling** if for each point $\epsilon_i \in \mathcal{S}_{n,\epsilon}$, the labels of the strands connected to i increase when one reads through them clockwise.

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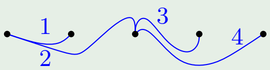

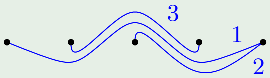

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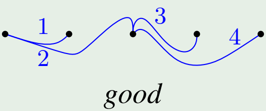

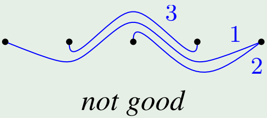

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Let $\mathcal{D}_{n,\epsilon}(k)$ denote the set of labeled strand diagrams on $\mathcal{S}_{n,\epsilon}$ with k strands and with good strand labelings.
Let $\mathcal{E}_\epsilon(k) := \{\text{exceptional sequences of length } k\}$.

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$$\begin{array}{rcl} \mathcal{E}_\epsilon(k) & \xrightarrow{\tilde{\Phi}} & \mathcal{D}_{n,\epsilon}(k) \\ \xi_\epsilon = (X_{i_1,j_1}, \ldots, X_{i_k,j_k}) & \longmapsto & \{(c(i_\ell, j_\ell), k+1-\ell)\}_{\ell \in [k]}. \end{array}$$

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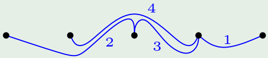

Change of setting

We now allow Q to be any quiver without loops or 2-cycles.

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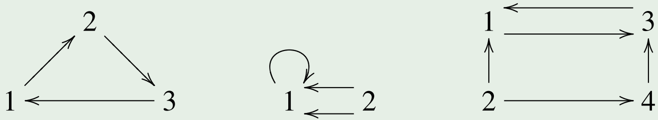

Shifting setting

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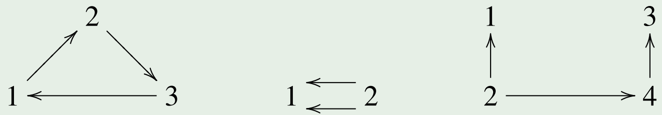
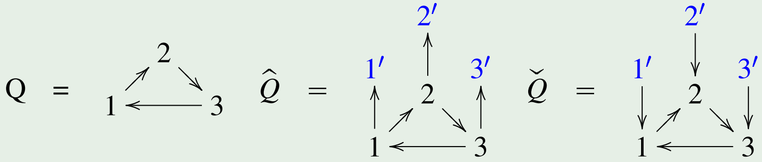


Definition

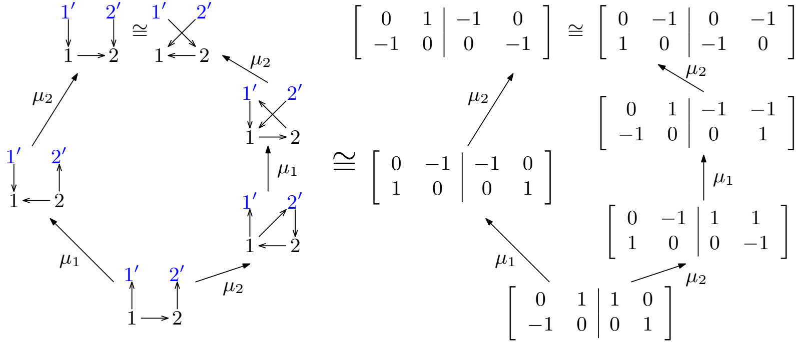
Given a quiver Q without loops or 2-cycles, the **framed quiver** (resp. **coframed quiver**) of Q, denoted $\widehat{Q}$ (resp. $\check{Q}$), is formed by

1. adding a **frozen vertex** i' for each vertex i in Q
2. adding an arrow $i \to i'$ (resp. $i \leftarrow i'$) for each vertex i in Q.

Example



The quiver $\widehat{Q}$ is an **ice quiver** and has vertices $\underbrace{[n]}_{\textbf{mutable}} \sqcup \underbrace{[n']}_{\textbf{frozen}}$.



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1 2
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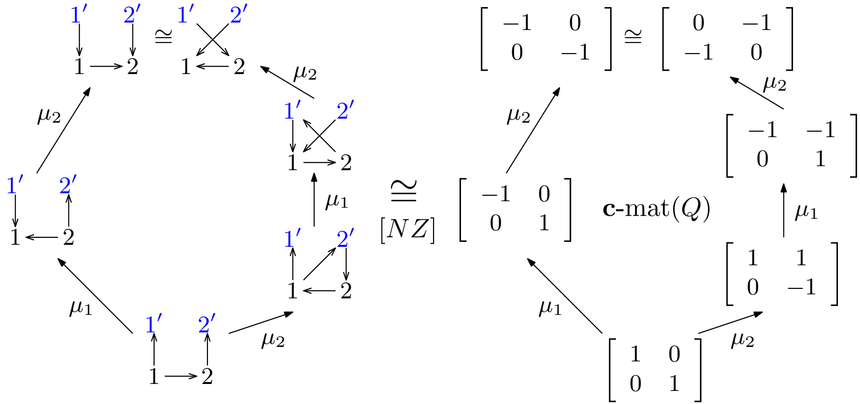

The row vectors of $C \in \mathbf{c}\text{-mat}(Q)$ are called **c-vectors**.

Theorem ("Sign-coherence" Derksen–Weyman–Zelevinsky 2008)

Any $\boldsymbol{c}$*-vector* $\overrightarrow{c}$ *is a nonzero element of* $\mathbb{Z}^n_{\geqslant 0}$ *or* $\mathbb{Z}^n_{\leqslant 0}$.

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Theorem (Chavez 2013)

Let Q *be acyclic. If* $\overrightarrow{c}$ *is a* $\mathbf{c}$*-vector appearing in some* $C \in \mathbf{c}\text{-}mat(Q)$*, then there exists an exceptional representation* $V \in rep_{\Bbbk}(Q)$ *such that* $|\overrightarrow{c}| = \underline{\dim}(V)$.

Notation

Let $\overrightarrow{c}$ *be a* $\mathbf{c}$*-vector of an acyclic quiver* Q*. Define*

$$|\overrightarrow{c}| \quad := \quad \begin{cases} \overrightarrow{c} & : \ \text{if } \overrightarrow{c} \text{ is positive} \\ -\overrightarrow{c} & : \ \text{if } \overrightarrow{c} \text{ is negative.} \end{cases}$$

Theorem (Speyer–Thomas 2013)

Let $C \in \mathbf{c}\text{-}mat(Q)$, let $\{\overrightarrow{c_i}\}_{i\in[n]}$ denote its **c***-vectors, and let $|\overrightarrow{c_i}| = \underline{\dim}(V_i)$ for some $V \in ind(rep_{\Bbbk}(Q))$. There exists a permutation $\sigma \in \mathfrak{S}_n$ such that*

$$(\underbrace{V_{\sigma(1)}, \ldots, V_{\sigma(j)}}_{-}, \underbrace{V_{\sigma(j+1)}, \ldots, V_{\sigma(n)}}_{+})$$

is a CES, and $Hom_{\Bbbk Q}(V_i, V_j) = 0$ if $\overrightarrow{c_i}, \overrightarrow{c_j}$ have the same sign. Conversely, any set of n vectors $\{\overrightarrow{c_i}\}_{i\in[n]}$ having these properties defines a **c***-matrix whose rows are $\{\overrightarrow{c_i}\}_{i\in[n]}$.*

Idea: **c**-matrices are complete exceptional collections with certain properties.

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Example

Let $\epsilon = (+,+,-,+,+)$ so that $Q = 1 \xrightarrow{+} 2 \xleftarrow{-} 3 \xrightarrow{+} 4$. Then $\mu_3 \circ \mu_2 \left(\widehat{Q}\right)$ has the following **c**-matrix and diagram.

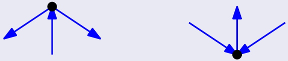
$$C = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$\overrightarrow{d}_C$

Theorem (G.–Igusa–Matherne–Ostroff)

Let $\overrightarrow{\mathcal{D}}_{n,\epsilon}$ denote the set of oriented diagrams $\overrightarrow{d} = \{\overrightarrow{c}(i_\ell, j_\ell)\}_{\ell\in[n]}$ on $\mathcal{S}_{n,\epsilon}$ with the property that any oriented subdiagram $\overrightarrow{d}_1$ of $\overrightarrow{d}$ consisting only of oriented strands connected to ϵ_k in $\mathcal{S}_{n,\epsilon}$ for some $k \in [0, n]$ is a subdiagram of one of the following:

- $\{\overrightarrow{c}(k, i_1), \overrightarrow{c}(k, i_2), \overrightarrow{c}(j, k)\}$ *where* $i_1 < k < i_2$ *and* $\epsilon_k = +$,
- $\{\overrightarrow{c}(i_1, k), \overrightarrow{c}(i_2, k), \overrightarrow{c}(k, j)\}$ *where* $i_1 < k < i_2$ *and* $\epsilon_k = -$.

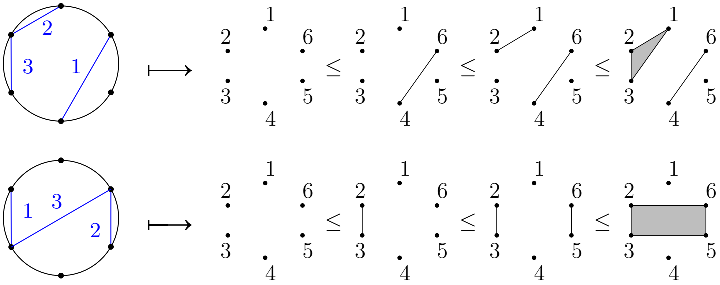


$\epsilon_k = +$ $\qquad$ $\epsilon_k = -$

*Then a matrix C belongs to **c**-mat(Q) if and only if $\overrightarrow{d}_C \in \overrightarrow{\mathcal{D}}_{n,\epsilon}$.*

Theorem (G.–Igusa–Matherne–Ostroff)

Exceptional sequences of $Q = 1 \leftarrow \cdots \leftarrow n$ *are in bijection with* ***saturated*** *chains in* $NC(\mathfrak{S}_{n+1})$*, the lattice of noncrossing partitions, that contain its bottom element* $\{\{i\}\}_{i\in[n+1]}$.



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