Triangulations of the continuous cluster category $\mathcal{C}_\pi$

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1. The construction of $\mathcal{C}_\pi$
   - $\mathcal{P}_{S^1}$: finitely generated projective representations of S^1
   - The Frobenius category $\mathcal{F}_\pi$
   - Stabilizing $\mathcal{F}_\pi$ to get $\mathcal{C}_\pi$ and the resulting topology
2. Quivers with multiplicity
   - The topological categories μ_σ
   - Continuous automorphisms τ of topological categories μ_σ
3. Triangulations of $\mathcal{C}_\pi$ and the minimal examples
   - (μ_σ, τ) as continuously triangulated coverings of $\mathcal{C}_\pi$
   - Triangulations of 2-sheeted covers of $\mathcal{C}_\pi$

Goals of this talk

- Review the construction of the continuous cluster category $\mathcal{C}_\pi$.
- Describe a classification of triangulations of $\mathcal{C}_\pi$.
- Exhibit the three possible ”minimal” examples

Motivation

Why study the continuous cluster category $\mathcal{C}_\pi$?

- Generalize cluster categories of type A_n
- General geometric interest: the disk model of the hyperbolic plane

Representations of S^1

Throughout this talk, let $S^1 = \mathbb{R}/2\pi\mathbb{Z}$ and R be a discrete valuation ring with uniformizer t and residue field $\overline{K} = K = R/(t)$, $char(K) \neq 2$.

Definition

A representation V of S^1 over R is given by an R-module $V[x]$ for each $[x] \in S^1$ and linear maps $V^{(x,\alpha)} : V[x] \to V[x-\alpha]$ for all $[x] \in S^1$ and $\alpha \in \mathbb{R}_{\geq 0}$ satisfying:

- $V^{(x-\beta,\alpha)} \circ V^{(x,\beta)} = V^{(x,\alpha+\beta)}$
- $V^{(x,2\pi n)} : V[x] \to V[x], m \mapsto t^n m, m \in V[x], \forall n \in \mathbb{N}$

Projectives representations of S^1

Definition

$P_{[x]}$ is a representation of S^1 given by $P_{[x]}[x-\alpha] := Re_x^\alpha$ for $\alpha \geq 0$ and unique R-linear homomorphisms $P_{[x]}^{(x-\alpha,\beta)} : P_{[x]}[x-\alpha] \to P_{[x]}[x-\alpha-\beta]$ defined by $P_{[x]}^{(x-\alpha,\beta)}(e_x^\alpha) = e_x^{\alpha+\beta}$.

Proposition

$P_{[x]}$ is projective and indecomposable for all $[x] \in S^1$. Any indecomposable is isomorphic to $P_{[x]}$ for some $[x] \in S^1$.

The topology of $Ind\mathcal{P}_{S^1}$

Definition

By topological R-category, we mean a small category whose object and morphism sets are topological spaces and whose structure maps are continuous, including the R-module structure maps of the hom-sets.

Example: $Ind\mathcal{P}_{S^1}$ is a topological category: $Ob(Ind\mathcal{P}_{S^1}) \simeq S^1$ and $Mor(Ind\mathcal{P}_{S^1}) = \{(r,x,y) | x \leq y \leq x + 2\pi\}/\sim$, where $\sim$ is defined by

- $(r,x,y) \sim (r, x+2\pi, y+2\pi), n \in \mathbb{Z}$
- $(r,x,x+2\pi) \sim (tr, x, x)$

The morphism (r,x,y) is defined by $e_x \mapsto re_y^{y-x}$.

Constructing $\mathcal{F}_\pi$ from $\mathcal{P}_{S^1}$

Definition

$\mathcal{F}_\pi$ is a category with objects given by pairs (V, d), where $V \in \mathcal{P}_{S^1}$ and d is an endomorphism with $d^2 = t$, and morphisms are $f : (V, d) \to (W, d')$ with $fd = d'f$.

Theorem

$\mathcal{F}_\pi$ is a Frobenius category.

Exact sequences of $\mathcal{F}_\pi$:
$(X, d) \xrightarrow{f} (Y, d') \xrightarrow{g} (Z, d'') \Leftrightarrow 0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ is split exact in $\mathcal{P}_{S^1}$.

Indecomposable and projective-injectve objects of $\mathcal{F}_\pi$

Proposition

1. $V \in \mathcal{P}_{S^1}$. *Let* $V^2 = \left(V \oplus V, \begin{bmatrix} 0 & t \\ 1 & 0 \end{bmatrix} \right)$. *Then* V^2 *is projective-injective.*
2. $\mathcal{F}_\pi$ *Krull-Schmidt.*
3. $\forall [x], [y] \in S^1$, $E(x, y) = \left(P_{[x]} \oplus P_{[y]}, \begin{bmatrix} 0 & \beta_* \\ \alpha_* & 0 \end{bmatrix} \right)$ *is indecomposable.*

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Represent $[x]$ and $[y]$ by reals satisfying $x \leq y \leq x + 2\pi$. Let $\alpha = y - x$ and $\beta = x + 2\pi - y$, giving morphisms $\alpha_* : P_{[x]} \leftrightarrow P_{[y]} : \beta_*$ given by $\alpha_*(e_x) = e_y^\alpha$ and $\beta_*(e_y) = e_x^\beta$.

The topology of the stable category $\underline{\mathcal{F}_\pi}$

- In the standard construciton, $Ob(Ind\mathcal{F}_\pi)$ has the topology of a Moebius band, and the projective-injective objects residing on the boundary of the band.
- This topology is preserved when we pass to the stable category $\underline{\mathcal{F}_\pi}$, except that the boundary is excluded.
- We may vary the construction of $Ind\mathcal{F}_\pi$ to be an even sheeted cover of the Moebius band.

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Proof Sketch: one object in each isomorphism class of $Ind\mathcal{F}_\pi \Rightarrow X = \tau(X)$. Also, there is a continuous never zero path from morphisms id_X and $f : X \to Y \Rightarrow \tau$ must be the identity functor.

Covers of $\mathcal{F}_\pi$

We must pass to (at least) a 2-sheeted cover of the Moebius band. In fact...

Theorem (G-Igusa)

Any cover of Ind$\mathcal{F}_\pi$ with an odd number of sheets does not admit a continuous triangulation.

Constructing $\mathcal{C}_\pi$

Definition

The continuous cluster category $\mathcal{C}_\pi$ is the additive closure of the category with objects ordered pairs $X = (x_0, x_1) \in (S^1)^2$ with $x_0 < x_1 < x_0 + 2\pi$. $\mathcal{C}_\pi(X, Y) = K$, when either
$x_0 \leq y_0 < x_1 \leq y_1 < x_0 + 2\pi$ or
$x_0 \leq y_1 < x_1 \leq y_0 + 2\pi < x_0 + 2\pi$ and 0 otherwise.

- $\mathcal{C}_\pi$ isomorphic with stabilization of the additive closure of any $2m$-sheeted cover of $Ind\mathcal{F}_\pi$.
- Clusters in $\mathcal{C}_\pi$ are given by maximal discrete laminations of the hyperbolic plane (i.e. a family of non-crossing geodesics such that each has ”its own neighborhood”).

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- Note KQ_n is **not basic**, meaning it has simple modules of dimension greater than 1.
- By adding structure to $\mathcal{C}_n$, we will construct triangulations of $\mathcal{C}_\pi$

Automorphisms of $\mathcal{C}_n$

- A set of bases $\{xij\}_{i,j\in[n]}$ is multiplicative if $x_{ij}x_{jk} = x_{ik}$.
- Any other set of bases $\{x'_{ij} = a_{ij}x_{ij}\}$, $a_{ij} \in K^*$, is multiplicative $\Leftrightarrow$ $a_{ij}a_{jk} = a_{ik}$. This is a multiplicative system of scalars.

Definition

An automorphism $\sigma : \mathcal{C}_n \to \mathcal{C}_n$ is a K-linear functor that

- on objects $\sigma \in S_n$
- on morphisms is a multiplicative system of scalars $\{a_{ij}\}$ such that on basic morphisms $\sigma(x_{ij}) = a_{ij}x_{\sigma(i)\sigma(j)}$.

Automorphisms of $\mathcal{C}_n$

Theorem

$\sigma \in Aut(\mathcal{C}_n)$. There is a set of multiplicative bases $\{x_{ij}\}$ so

- *$a_{ij} = 1$ is i and j are in the same σ-orbit, and*
- *$a_{ik} = a_{jl}$ if i and j are in the same σ-orbit and k and l are in the same σ-orbit.*

We call a set of bases such as in the above theorem good bases.

Automorphisms of $(\mathcal{C}_n, \sigma)$

Fix $\sigma \in Aut(\mathcal{C}_n)$ and a set of good bases $\{x_{ij}\}$ with respect to σ. An automorphism τ of the pair $(\mathcal{C}_n, \sigma)$ is specified as follows:

- $\tau \in S_n$ for objects of $\mathcal{C}_n$
- $\tau(x_{ij}) = b_{ij} x_{\tau(i)\tau(j)}$ with b_{ij} a multiplicative system
- $\tau\sigma = \sigma\tau$. In terms of coefficients, $b_{\sigma(i)\sigma(j)} a_{ij} = a_{\tau(i)\tau(j)} b_{ij}$.

$\{b_{ij}\}$ are called the transition factors of τ. In general, it is not possible to find a set of bases which are good for both σ and τ.

General quivers with multiplicity

In general,

- Q a quiver with p vertices; $Q_{\underline{n}}$ has multiplicity n_i at vertex i, where $\underline{n} = (n_1, \ldots, n_p)$.
- Again, $KQ_{\underline{n}}$ is not basic.
- Fixing good bases $\{x_{ij}\}$ for $(\mathcal{C}_{n_r}, \sigma_r)$ and $\{y_{kl}\}$ for $(\mathcal{C}_{n_s}, \sigma_s)$, we can write down conditions for the transition factors of $\alpha : (\mathcal{C}_{n_r}, \sigma_r) \to (\mathcal{C}_{n_s}, \sigma_s)$ ensuring continuity.

Clutching map and μ_σ

- Given σ an automorphism of $\mathcal{C}_n$, let $\mathcal{R} = \{(x, y)|x \leq y \leq x + 2\pi\} \subset \mathbb{R}^2$. Then

$$\mu_\sigma = (\mathcal{R} \times \mathcal{C}_n)/\sigma_*$$

  where $\sigma_*(y, x + 2\pi, i) = (x, y, \sigma(i))$.
- This is a covering category for the category of the open Moebius band.
- σ plays the role of a clutching map for the bundle that is an n-sheeted cover of the Moebius band over S^1.

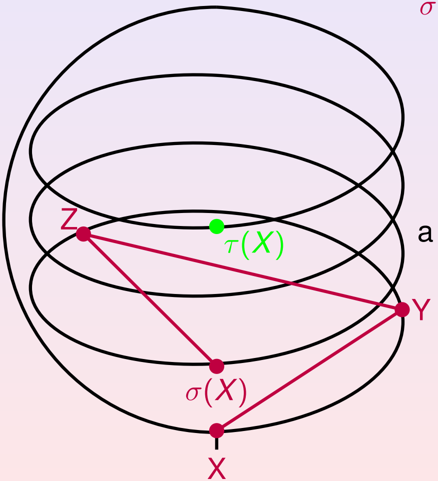
Continuous automorphisms τ

- σ determines the topology of μ_σ, so τ an automorphism of $(\mathcal{C}_n, \sigma)$ determines a continuous automorphism of the category μ_σ, $\tau^*(x, y, i) = (y, x, \tau(i))$.
- μ_σ is algebraically equivalent to $\mathcal{C}_\pi$ and τ will play the role of a shift functor for μ_σ.
- We need a way to specify distinguished triangles with respect to this triangulation functor to get a triangulation of $\mathcal{C}_\pi$.

Example

$\sigma = (1234) \in Aut(\mathcal{C}_4)$

$\tau = \sigma^3 = (2341)$



a schematic of (μ_σ, τ)

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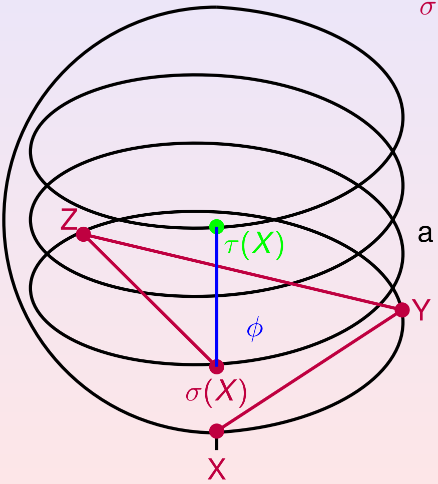

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$\phi : \sigma \to \tau$

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The natural isomorphism ϕ for distinguished triangles

ϕ a K-linear, so $\phi_i = \cdot c_i$ for some $c_i \in K^*$. These constants are subject to some relations with the multiplicative system $\{a_{ij}\}$ of σ and $\{x_{ij}\}$, and the transition factors $\{b_{ij}\}$ of τ:

$$b_{ij}c_j = c_i a_{ij}\ ,\ c_{\sigma(i)} = -c_i a_{\sigma^{-1}\tau(i),i}$$

We define triangles in μ_σ as follows:

$$(x, y, i) \xrightarrow{1} (x, z, i) \xrightarrow{1} (y, z, i) \rightarrow (y, x, \tau(i))$$

where the last arrow is the composition

$$(y, z, i) \xrightarrow{1} (y, x + 2\pi, i) \sim (x, y, \sigma(i)) \xrightarrow{\phi} (y, x, \tau(i))$$

Classifying triangulations

Theorem

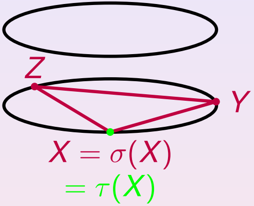
There is a continuous triangulation of $\mathcal{C}_\pi$ for the data

- *an even integer n*
- *pair of commuting automorphisms of $\mathcal{C}_n$, σ and τ*
- *natural transformation $\phi : \sigma \to \tau$ satisfying the conditions above.*
- *$\sigma(i)$ and $\tau(i)$ cannot reside in the same odd cycle of σ*

All algebraically triangulated coverings of $\mathcal{C}_\pi$ arise in this way.

The case of 2 sheeted covers

not a triangulation

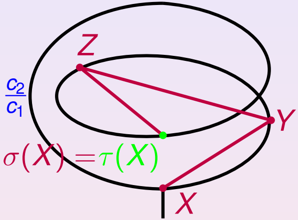


$\sigma = id$

$\tau = id$

$\phi : a_{12} = -\frac{c_2}{c_1}$

Igusa-Todorov



$\sigma = (12)$

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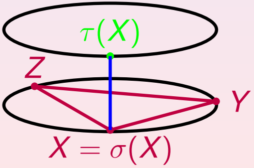
Orlov

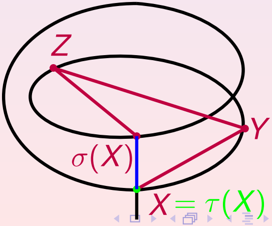


$\sigma = id$

$\tau = (12)$

$\phi : b_{12} = -\frac{c_2}{c_1}$

a third triangulation



$\sigma = (12)$

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The known triangulations

- Igusa-Todorov appears in ”Continuous Frobenius Categories” arXiv:1209.0038v3 [math.RT] 21 Jan 2013
- Orlov appears in ”Landau-Ginzburg Models, D-branes and Mirror Symmetry” arXiv.1111.2962v1 [math.AG] 12 Nov 2011

Closing remarks

- A very coarse classification: what's isomorphic, or triangle equivalent?
- Generalize to cluster categories of infinite rank not of type *A*, "composition relations"
- Key to understanding bundles of cluster categories of surface type?