Triangulations of the continuous cluster category $C_\pi$

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joint work with Kiyoshi Igusa

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1. The construction of $C_\pi$
   - $\mathcal{P}_{S^1}$: finitely generated projective representations of $S^1$
   - The Frobenius category $\mathcal{F}_\pi$
   - Stabilizing $\mathcal{F}_\pi$ to get $C_\pi$ and the resulting topology

2. Quivers with multiplicity
   - The topological categories $\mu_\sigma$
   - Continuous automorphisms $\tau$ of topological categories $\mu_\sigma$

3. Triangulations of $C_\pi$ and the minimal examples
   - $(\mu_\sigma, \tau)$ as continuously triangulated coverings of $C_\pi$
   - Triangulations of 2-sheeted covers of $C_\pi$
Outline
The construction of $C_\pi$
Quivers with multiplicity
Triangulations of $C_\pi$ and the minimal examples

Goals of this talk

- Review the construction of the continuous cluster category $C_\pi$.
- Describe a classification of triangulations of $C_\pi$.
- Exhibit the three possible "minimal" examples.
Motivation

Why study the continuous cluster category $C_\pi$?

- Generalize cluster categories of type $A_n$
- General geometric interest: the disk model of the hyperbolic plane
Throughout this talk, let \( S^1 = \mathbb{R}/2\pi \mathbb{Z} \) and \( R \) be a discrete valuation ring with uniformizer \( t \) and residue field \( \overline{K} = K = R/(t) \), \( \text{char}(K) \neq 2 \).

**Definition**

A representation \( V \) of \( S^1 \) over \( R \) is given by an \( R \)-module \( V[x] \) for each \( [x] \in S^1 \) and linear maps \( V(x,\alpha) : V[x] \to V[x-\alpha] \) for all \( [x] \in S^1 \) and \( \alpha \in \mathbb{R}_{\geq 0} \) satisfying:

- \( V(x-\beta,\alpha) \circ V(x,\beta) = V(x,\alpha+\beta) \)
- \( V(x,2\pi n) : V[x] \to V[x], m \mapsto t^n m, m \in V[x], \forall n \in \mathbb{N} \)
Projectives representations of $S^1$

**Definition**

$P_{[x]}$ is a representation of $S^1$ given by $P_{[x]}[x - \alpha] := Re_x^\alpha$ for $\alpha \geq 0$ and unique $R$-linear homomorphisms

$P_{[x]}^{(x-\alpha,\beta)} : P_{[x]}[x - \alpha] \to P_{[x]}[x - \alpha - \beta]$ defined by

$P_{[x]}^{(x-\alpha,\beta)}(e_x^\alpha) = e_x^{\alpha+\beta}$.

**Proposition**

$P_{[x]}$ is projective and indecomposable for all $[x] \in S^1$. Any indecomposable is isomorphic to $P_{[x]}$ for some $[x] \in S^1$. 

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Triangulations of the continuous cluster category $C_\pi$
The topology of $\text{Ind}\mathcal{P}_{S^1}$

Definition

By topological $R$-category, we mean a small category whose object and morphism sets are topological spaces and whose structure maps are continuous, including the $R$-module structure maps of the hom-sets.

Example: $\text{Ind}\mathcal{P}_{S^1}$ is a topological category: $\text{Ob}(\text{Ind}\mathcal{P}_{S^1}) \simeq S^1$ and $\text{Mor}(\text{Ind}\mathcal{P}_{S^1}) = \{(r, x, y) | x \leq y \leq x + 2\pi\}/\sim$, where $\sim$ is defined by

- $(r, x, y) \sim (r, x + 2\pi, y + 2\pi), \ n \in \mathbb{Z}$
- $(r, x, x + 2\pi) \sim (tr, x, x)$

The morphism $(r, x, y)$ is defined by $e_x \mapsto re^{y-x}$. 

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Triangulations of the continuous cluster category $C_\pi$
### Constructing $\mathcal{F}_\pi$ from $\mathcal{P}_{S^1}$

#### Definition

$\mathcal{F}_\pi$ is a category with objects given by pairs $(V, d)$, where $V \in \mathcal{P}_{S^1}$ and $d$ is an endomorphism with $d^2 = t$, and morphisms are $f : (V, d) \to (W, d')$ with $fd = d'f$.

#### Theorem

$\mathcal{F}_\pi$ is a Frobenius category.

Exact sequences of $\mathcal{F}_\pi$:

$(X, d) \xrightarrow{f} (Y, d') \xrightarrow{g} (Z, d'') \iff 0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ is split exact in $\mathcal{P}_{S^1}$.
Indecomposable and projective-injective objects of $\mathcal{F}_\pi$

**Proposition**

1. $V \in \mathcal{P}_{S^1}$. Let $V^2 = \left( V \oplus V, \begin{bmatrix} 0 & t \\ 1 & 0 \end{bmatrix} \right)$. Then $V^2$ is projective-injective.

2. $\mathcal{F}_\pi$ Krull-Schmidt.

3. $\forall [x], [y] \in S^1$, $E(x, y) = \left( P_{[x]} \oplus P_{[y]}, \begin{bmatrix} 0 & \beta_* \\ \alpha_* & 0 \end{bmatrix} \right)$ is indecomposable.
Proposition

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2. \( \mathcal{F}_\pi \) is Krull-Schmidt.

3. \( \forall [x], [y] \in S^1, E(x, y) = \left( P[x] \oplus P[y], \begin{bmatrix} 0 & \beta_* \\ \alpha_* & 0 \end{bmatrix} \right) \) is indecomposable.

Represent \([x] \) and \([y] \) by reals satisfying \( x \leq y \leq x + 2\pi \). Let \( \alpha = y - x \) and \( \beta = x + 2\pi - y \), giving morphisms \( \alpha_* : P[x] \hookrightarrow P[y] : \beta_* \) given by \( \alpha_*(e_x) = e_y^\alpha \) and \( \beta_*(e_y) = e_x^\beta \).

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Triangulations of the continuous cluster category \( C_\pi \)
The topology of the stable category $\mathcal{F}_\pi$

- In the standard construction, $\text{Ob}(\text{Ind}\mathcal{F}_\pi)$ has the topology of a Möbius band, and the projective-injective objects residing on the boundary of the band.

- This topology is preserved when we pass to the stable category $\mathcal{F}_\pi$, except that the boundary is excluded.

- We may vary the construction of $\text{Ind}\mathcal{F}_\pi$ to be an even sheeted cover of the Möbius band.
The topology of the stable category $\cal F_\pi$

$\text{Ind}\, \cal F_\pi$ usually has a single object from each isomorphism class. However...
The topology of the stable category $\mathcal{F}_\pi$

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**Theorem (Igusa-Todorov)**

*There is no way to continuously triangulate the stable category of $\text{add}(\text{Ind}\mathcal{F}_\pi)$ preserving the topology of the subcategory of indecomposables as an open Moebius band.*
The topology of the stable category $\mathcal{F}_\pi$

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**Theorem (Igusa-Todorov)**

There is no way to continuously triangulate the stable category of $\text{add}(\text{Ind}\mathcal{F}_\pi)$ preserving the topology of the subcategory of indecomposables as an open Moebius band.

Proof Sketch: one object in each isomorphism class of $\text{Ind}\mathcal{F}_\pi \Rightarrow X = \tau(X)$. Also, there is a continuous never zero path from morphisms $id_X$ and $f : X \rightarrow Y \Rightarrow \tau$ must be the identity functor.
Covers of $\mathcal{F}_\pi$

We must pass to (at least) a 2-sheeted cover of the Moebius band. In fact...

Theorem (G-Igusa)

Any cover of $\text{Ind}\mathcal{F}_\pi$ with an odd number of sheets does not admit a continuous triangulation.
Constructing $C_\pi$

**Definition**

The **continuous cluster category** $C_\pi$ is the additive closure of the category with objects ordered pairs $X = (x_0, x_1) \in (S^1)^2$ with $x_0 < x_1 < x_0 + 2\pi$. $C_\pi(X, Y) = K$, when either

- $x_0 \leq y_0 < x_1 \leq y_1 < x_0 + 2\pi$ or
- $x_0 \leq y_1 < x_1 \leq y_0 + 2\pi < x_0 + 2\pi$ and 0 otherwise.

- $C_\pi$ isomorphic with stabilization of the additive closure of any $2m$-sheeted cover of $IndF_\pi$.
- Clusters in $C_\pi$ are given by maximal discrete laminations of the hyperbolic plane (i.e. a family of non-crossing geodesics such that each has ”its own neighborhood”).
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That was complicated. Let’s start again as simply as possible. Let $Q = A_1$:

$\cdot A_1$
The categories $C_n$

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- Let $C_n := \text{Ind}(\text{mod-}KQ_n)$, having $n$ isomorphic objects be a Schurian $K$-category with $C_n(j, i) = Kx_{ij}$. 
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- $Q_n$ is the quiver with multiplicity $n$, in the present case just $n$ vertices with no arrows.
- Let $\mathcal{C}_n := Ind(\text{mod-}KQ_n)$, having $n$ isomorphic objects be a Schurian $K$-category with $\mathcal{C}_n(j, i) = Kx_{ij}$.
- Note $KQ_n$ is not basic, meaning it has simple modules of dimension greater than 1.
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Let $C_n := \text{Ind}(\text{mod-KQ}_n)$, having $n$ isomorphic objects be a Schurian $K$-category with $C_n(j, i) = Kx_{ij}$.

Note $KQ_n$ is not basic, meaning it has simple modules of dimension greater than 1.

By adding structure to $C_n$, we will construct triangulations of $C_\pi$. 
A set of bases \( \{x_{ij}\}_{i,j \in [n]} \) is multiplicative if \( x_{ij} x_{jk} = x_{ik} \).

Any other set of bases \( \{x'_{ij} = a_{ij} x_{ij}\}, a_{ij} \in \mathbb{K}^* \), is multiplicative \( \iff a_{ij} a_{jk} = a_{ik} \). This is a multiplicative system of scalars.

**Definition**

An automorphism \( \sigma : C_n \to C_n \) is a \( \mathbb{K} \)-linear functor that

- on objects \( \sigma \in S_n \)
- on morphisms is a multiplicative system of scalars \( \{a_{ij}\} \)
  such that on basic morphisms \( \sigma(x_{ij}) = a_{ij} x_{\sigma(i)\sigma(j)} \).
Automorphisms of $\mathcal{C}_n$

**Theorem**

$\sigma \in \text{Aut}(\mathcal{C}_n)$. There is a set of multiplicative bases $\{x_{ij}\}$ so

- $a_{ij} = 1$ if $i$ and $j$ are in the same $\sigma$-orbit, and
- $a_{ik} = a_{jl}$ if $i$ and $j$ are in the same $\sigma$-orbit and $k$ and $l$ are in the same $\sigma$-orbit.

We call a set of bases such as in the above theorem **good bases**.
Automorphisms of \((\mathcal{C}_n, \sigma)\)

Fix \(\sigma \in \text{Aut}(\mathcal{C}_n)\) and a set of good bases \(\{x_{ij}\}\) with respect to \(\sigma\). An automorphism \(\tau\) of the pair \((\mathcal{C}_n, \sigma)\) is specified as follows:

- \(\tau \in S_n\) for objects of \(\mathcal{C}_n\)
- \(\tau(x_{ij}) = b_{ij}x_{\tau(i)\tau(j)}\) with \(b_{ij}\) a multiplicative system
- \(\tau\sigma = \sigma\tau\). In terms of coefficients, \(b_{\sigma(i)\sigma(j)}a_{ij} = a_{\tau(i)\tau(j)}b_{ij}\).

\(\{b_{ij}\}\) are called the transition factors of \(\tau\). In general, it is not possible to find a set of bases which are good for both \(\sigma\) and \(\tau\).
In general,

- $Q$ a quiver with $p$ vertices; $Q_{\underline{n}}$ has multiplicity $n_i$ at vertex $i$, where $\underline{n} = (n_1, \ldots, n_p)$.
- Again, $KQ_{\underline{n}}$ is not basic.
- Fixing good bases $\{x_{ij}\}$ for $(C_{n_r}, \sigma_r)$ and $\{y_{kl}\}$ for $(C_{n_s}, \sigma_s)$, we can write down conditions for the transition factors of $\alpha: (C_{n_r}, \sigma_r) \rightarrow (C_{n_s}, \sigma_s)$ ensuring continuity.
Clutching map and $\mu_\sigma$

- Given $\sigma$ an automorphism of $C_n$, let
  \[ \mathcal{R} = \{(x, y) | x \leq y \leq x + 2\pi \} \subset \mathbb{R}^2. \]  
  Then
  \[ \mu_\sigma = (\mathcal{R} \times C_n)/\sigma_*. \]

- where $\sigma_*(y, x + 2\pi, i) = (x, y, \sigma(i))$.

- This is a covering category for the category of the open Moebius band.

- $\sigma$ plays the role of a clutching map for the bundle that is an $n$-sheeted cover of the Moebius band over $S^1$. 

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Continuous automorphisms $\tau$

- $\sigma$ determines the topology of $\mu_\sigma$, so $\tau$ an automorphism of $(C_n, \sigma)$ determines a continuous automorphism of the category $\mu_\sigma$, $\tau^*(x, y, i) = (y, x, \tau(i))$.
- $\mu_\sigma$ is algebraically equivalent to $C_\pi$ and $\tau$ will play the role of a shift functor for $\mu_\sigma$.
- We need a way to specify distinguished triangles with respect to this triangulation functor to get a triangulation of $C_\pi$.
Example

\[ \sigma = (1234) \in Aut(C_4) \]
\[ \tau = \sigma^3 = (2341) \]

A schematic of \((\mu_\sigma, \tau)\)
Example

\[ \sigma = (1234) \in \text{Aut}(C_4) \]
\[ \tau = \sigma^3 = (2341) \]
\[ \phi : \sigma \to \tau \]

A schematic of \((\mu_\sigma, \tau)\)
The natural isomorphism $\phi$ for distinguished triangles

$\phi$ a $K$-linear, so $\phi_i = \cdot c_i$ for some $c_i \in K^*$. These constants are subject to some relations with the multiplicative system $\{a_{ij}\}$ of $\sigma$ and $\{x_{ij}\}$, and the transition factors $\{b_{ij}\}$ of $\tau$:

$$b_{ij}c_j = c_i a_{ij}, \quad c_{\sigma(i)} = -c_i a_{\sigma^{-1} \tau(i), i}$$

We define triangles in $\mu_{\sigma}$ as follows:

$$(x, y, i) \xrightarrow{1} (x, z, i) \xrightarrow{1} (y, z, i) \rightarrow (y, x, \tau(i))$$

where the last arrow is the composition

$$(y, z, i) \xrightarrow{1} (y, x + 2\pi, i) \sim (x, y, \sigma(i)) \xrightarrow{\phi} (y, x, \tau(i))$$
Theorem

There is a continuous triangulation of $C_\pi$ for the data

- an even integer $n$
- pair of commuting automorphisms of $C_n$, $\sigma$ and $\tau$
- natural transformation $\phi : \sigma \rightarrow \tau$ satisfying the conditions above.
- $\sigma(i)$ and $\tau(i)$ cannot reside in the same odd cycle of $\sigma$

All algebraically triangulated coverings of $C_\pi$ arise in this way.
The case of 2-sheeted covers

Not a triangulation

\[ X = \sigma(X) = \tau(X) \]

\[ \sigma = \text{id} \]
\[ \tau = \text{id} \]
\[ \phi : a_{12} = -\frac{c_2}{c_1} \]

Igusa-Todorov

\[ \sigma(X) = \tau(X) \]
\[ \sigma = (12) \]
\[ \tau = (12) \]
\[ \phi : c_2 = -c_1 \]

Orlov

\[ X = \sigma(X) = \tau(X) \]
\[ \sigma = \text{id} \]
\[ \tau = (12) \]
\[ \phi : b_{12} = -\frac{c_2}{c_1} \]

A third triangulation

\[ \sigma(X) = \tau(X) \]
\[ \sigma = (12) \]
\[ \tau = \text{id} \]
\[ \phi : c_2 = -c_1 \]
The known triangulations

Closing remarks

- A very coarse classification: what’s isomorphic, or triangle equivalent?
- Generalize to cluster categories of infinite rank not of type $A$, ”composition relations”
- Key to understanding bundles of cluster categories of surface type?