Syzygies over 2-Calabi Yau tilted algebras

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PART 1: Definitions and general results

- 2-CY tilted algebras
- d-Gorenstein algebras
- Results

Part 2: 2-CY tilted algebras arising from surfaces

- Unpunctured case
- Punctured disc

2-Calabi Yau tilted algebras

Let k be an algebraic closed field. A k-linear Hom-finite triangulated category $\mathcal{C}$ with suspension functor [1] is 2-Calabi Yau (2-CY) if there is a functorial isomorphism $D\mathrm{Ext}^1_{\mathcal{C}}(X, Y) \simeq \mathrm{Ext}^1_{\mathcal{C}}(Y, X)$, for X, Y in $\mathcal{C}$.

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A k-linear subcategory $\mathcal{T}$ of $\mathcal{C}$ is cluster tilting if $\mathrm{Ext}^1_{\mathcal{C}}(T,T') = 0$ for all $T, T' \in \mathcal{T}$, and if there is an $X \in \mathcal{C}$ such that $\mathrm{Ext}^1_{\mathcal{C}}(X,T) = 0$ for all $T \in \mathcal{T}$, then $X \in \mathcal{T}$.

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The endomorphism algebra $\mathrm{B} = \text{End}_{\mathcal{C}}(T)$ is called a 2-CY tilted algebra.

Properties

For each object X in $\mathcal{C}$ there are triangles

$$T_1 \to T_0 \to X \to T_1[1] \quad (1)$$

$$T_1'[1] \to X \to T_0'[2] \to T_1'[2] \quad (2)$$

where T_0, T_1, T_0', T_1' are in $\mathcal{T}$.

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If we denote by $(\mathcal{T}[1])$ the ideal of all morphisms which factor through an element in $\mathcal{T}[1]$, there is an equivalence [BMR07] [KR07].

$$F : \mathcal{C}/(\mathcal{T}[1]) \to \text{mod}\mathrm{B}$$
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Every 2-CY tilted algebra B is Gorenstein of dimension at most one [KR07].

d-Gorenstein algebras

A finite dimensional Artin algebra Λ is Gorenstein of dimension d (d-Gorenstein) if $d = \text{proj.dim}_{\Lambda} D(\Lambda_{\Lambda}) = \text{inj.dim}_{\Lambda}\Lambda < \infty$.

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$M \in \text{mod}\Lambda$ is projectively Cohen-Macaulay (CMP) if $\text{Ext}^i_{\Lambda}(M, \Lambda) = 0 \ \forall i > 0$.
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The category CMP(Λ) is a full exact subcategory of modΛ, it is Frobenius, the projective-injective objects are the projectives in modΛ. The stable category $\underline{\text{CMP}}(\Lambda)$ is triangulated, the inverse shift is given by the usual syzygy operator Ω. (Dual $\underline{\text{CMI}}(\Lambda)$ and Ω^{-1}) [Bu86].

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The AR translations act as triangle quasi-inverse equivalences [BR07].

$$\tau : \underline{\text{CMP}}(\Lambda) \rightleftarrows \underline{\text{CMI}}(\Lambda) : \tau^{-1}$$

Results

Theorem (GE-Schiffler)

For indecomposable modules M and N in a 2-CY tilted algebra B, the following statements are equivalent

(a1) M is a non projective syzygy,

(a2) M belongs to CMP(B),

(a3) $\Omega^2 \tau M \simeq M$,

(a4) $\Omega^{-2} M \simeq \tau M$.

(b1) N is a non injective co-syzygy,

(b2) N belongs to CMI(B),

(b3) $\Omega^{-2} \tau^{-1} N \simeq N$,

(b4) $\Omega^2 N \simeq \tau^{-1} N$.

Corollary

The objects in CMP(B) are the non projective syzygies on mod$\mathrm{B}$. The objects in CMI(B) are the non injective co-syzygies on mod$\mathrm{B}$.

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(a4) $\Omega^{-2}M \simeq \tau M$.

(b1) N is a non injective co-syzygy,

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(b3) $\Omega^{-2}\tau^{-1}N \simeq N$,

(b4) $\Omega^2 N \simeq \tau^{-1}N$.

Corollary

The objects in $\underline{\mathrm{CMP}}(\mathrm{B})$ are the non projective syzygies on mod$\mathrm{B}$. The objects in $\underline{\mathrm{CMI}}(\mathrm{B})$ are the non injective co-syzygies on mod$\mathrm{B}$.

If Λ is a d-Gorenstein Artin algebra then the objects in $\underline{\mathrm{CMP}}(\Lambda)$ are the d-th non projective syzygies on modΛ. [Bel00].

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- The objects in $\underline{\mathrm{CMP}}(\mathrm{B})$ are the non projective syzygies on mod$\mathrm{B}$.
- If B is a tame cluster tilted algebra, then $\underline{\mathrm{CMP}}(\mathrm{B})$ has a finite number of indecomposable modules. [BO11]

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- The objects in $\underline{\mathrm{CMP}}(\mathrm{B})$ are the non projective syzygies on mod$\mathrm{B}$.
- If B is a tame cluster tilted algebra, then $\underline{\mathrm{CMP}}(\mathrm{B})$ has a finite number of indecomposable modules. [BO11]
- If an Artin algebra A is torsionless finite (the number of indecomposable submodules of projective modules is finite) then rep.dim$A \leq 3$. [Rin11]

Results

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- they generalize the concept of projective dimension of a module
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Theorem (GE-Schiffler)

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Theorem (GE-Schiffler)

Let Λ be a d-Gorenstein artin algebra, then $\phi\text{dim}(\Lambda) = \psi\text{dim}(\Lambda) = d$.

Corollary

For every 2-CY tilted algebra B, $\phi\text{dim}(\mathrm{B}) = \psi\text{dim}(\mathrm{B}) \leq 1$.

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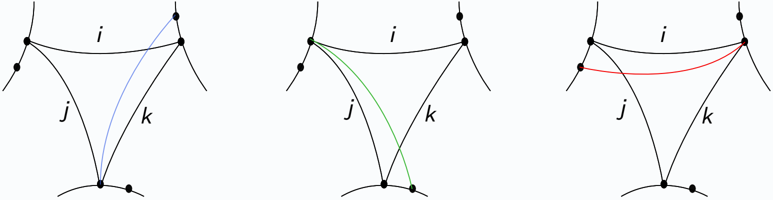
For gentle algebras, the only indecomposable non projective syzygies are summands on the radical of an indecomposable projective. [K14]

$$\cdots \longrightarrow P(i) \longrightarrow P(k) \longrightarrow P(j) \longrightarrow M(j{\to}\beta_j) \longrightarrow 0$$

$$M(j{\to}\beta_j) \hookrightarrow P(i) \twoheadrightarrow M(i{\to}\beta_i) \hookrightarrow P(k) \twoheadrightarrow M(k{\to}\beta_k) \hookrightarrow P(j)$$



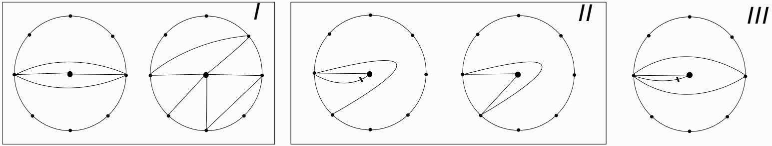
Punctured disc

The cluster category of type $\mathbb{D}_n$ has a geometric model given by the punctured disc with n marked points on the boundary [S08]. There are bijections:

Arcs (tagged arcs)	$\leftrightarrow$	Indecomposable objects in $\mathcal{C}_{\mathbb{D}_n}$
Triangulations	$\leftrightarrow$	Cluster tilting objects in $\mathcal{C}_{\mathbb{D}_n}$
Arcs $\notin$ triangulation	$\leftrightarrow$	Indecomposable modules in $\operatorname{mod}\mathrm{B}$
AR translation τ	$\leftrightarrow$	Clockwise rotation of angle $2\pi/n$ (change tags)

Punctured disc

We classify the triangulations in three types



- Type II: all the non projective syzygies arise from internal triangles with 3 vertices on the boundary (unpunctured case)
- Type III: all except 3 non projective syzygies arise from internal triangles with 3 vertices on the boundary (unpunctured case)

Punctured disc

For Type I, we define a color-index label for marked points in the boundary such that

Theorem (GE-Schiffler)

In a Type I triangulation, given $M(r_i, b_j)$ where $j \in \{i+2, ..., i-1\}$, then

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and we prove that in this case, all non projective syzygies

- arise from internal triangles with 3 vertices on the boundary, or
- belong to the family $M(r_i, b_j)$ described in the theorem

Punctured disc

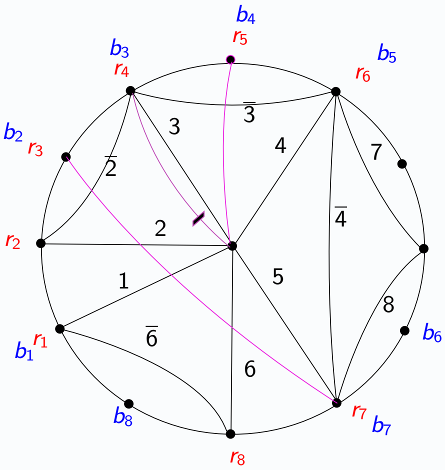

Figure: Type I triangulation with colored points. Curves associated to the modules $M(r_7, b_2)$, $M(r_5, b_4)$ and $M(r_4, b_3)$

- [*Bu*86] Maximal Cohen-Macaulay modules and Tate cohomology. R.O. Buchweitz.
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- [*BO*11] Cluster tilting and complexity. A. Bergh, S. Oppermann.
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- [*Kal*14] Singularity categories of gentle algebras. M. Kalck.

Thanks

(Thanks)^2