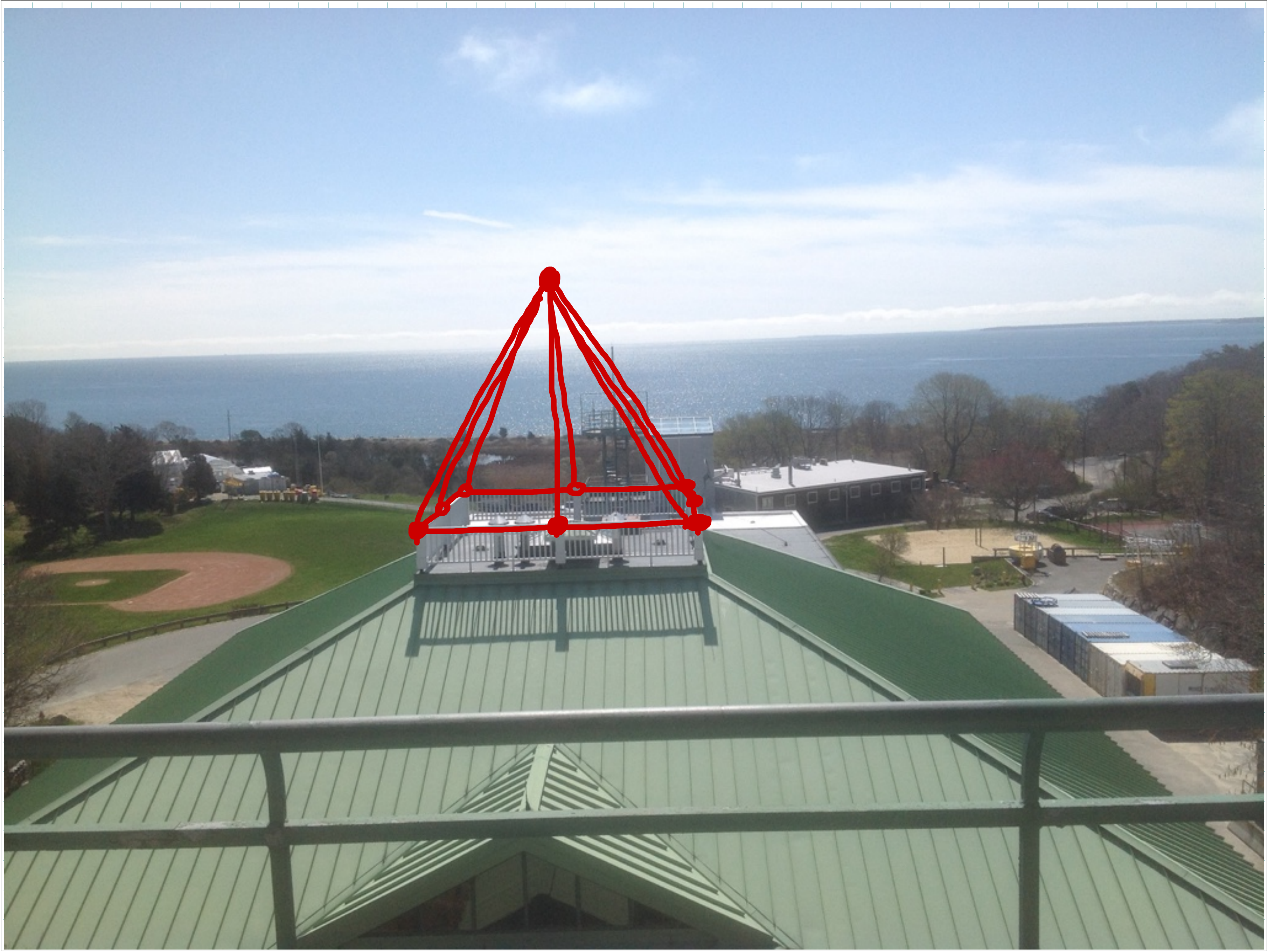
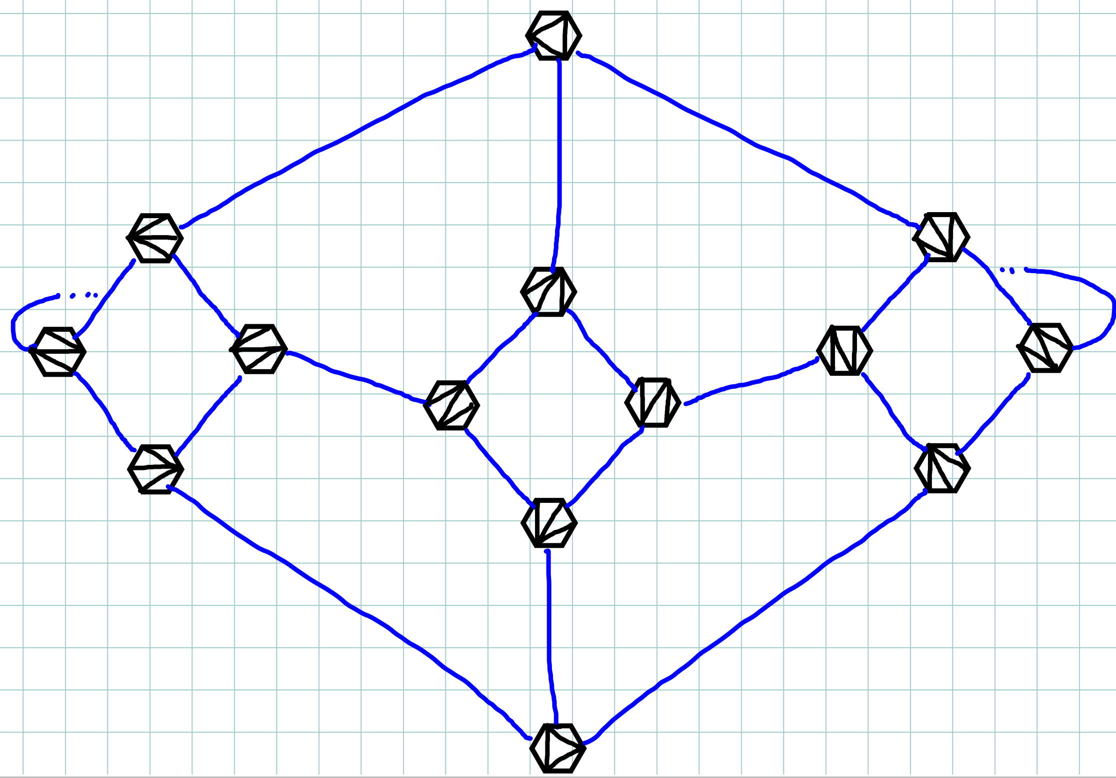
On the non-leaving-face property for associahedra

Thomas Brüstle



D. Sleator, R. Tarjan, W. Thurston, 1988:

The associahedron has the non-leaving-face property (NLF): every geodesic connecting two vertices in the graph stays in the minimal face containing both.



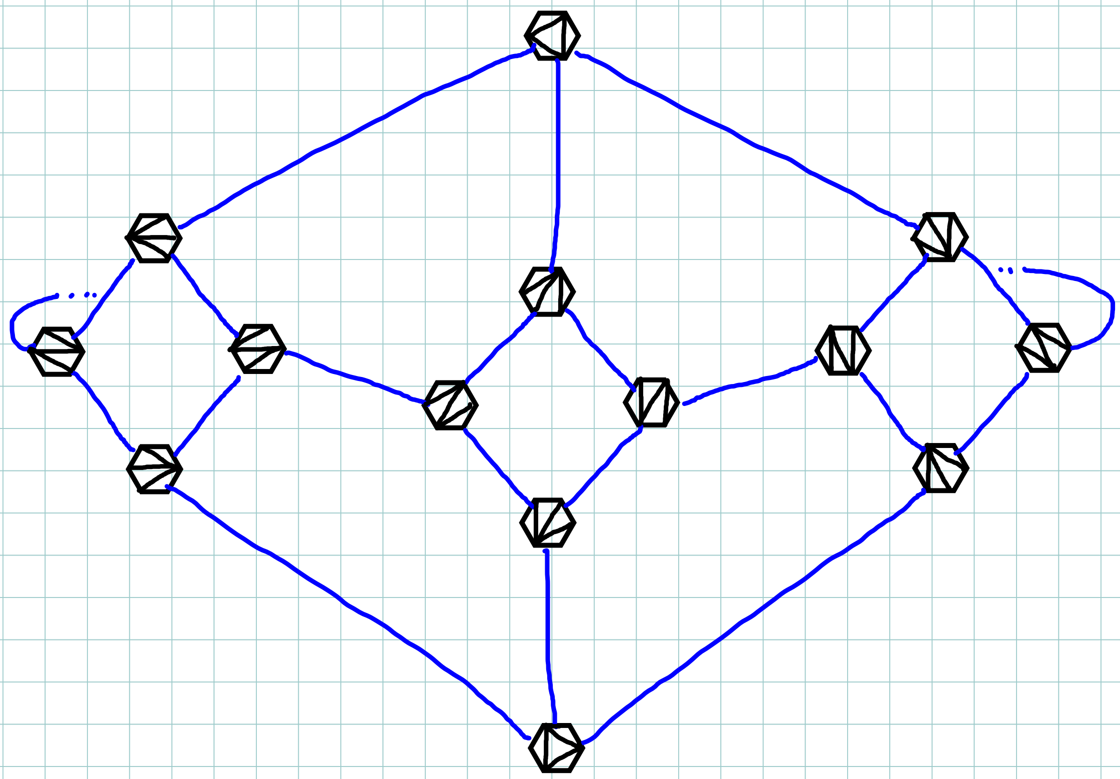
A_3

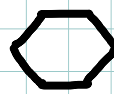
The associahedron (type A_n)

= Stasheff polytope
= Tamari lattice
= exchange graph of type A_n

The associahedron (type A_n)

Definition 1: vertices = triangulations of $(n+3)$-gon
edges = flips of diagonals



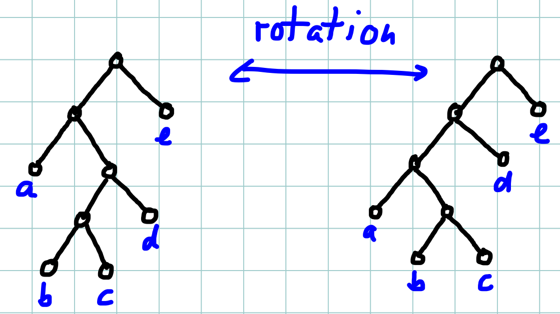
A_3 : 

hexagon

The associahedron (type A_n)

Definition 2: vertices = binary trees with $n+2$ leaves
edges = rotations

$n=3$:



$(a((bc)d))e$ $\quad$ $(a(bc))d)e$

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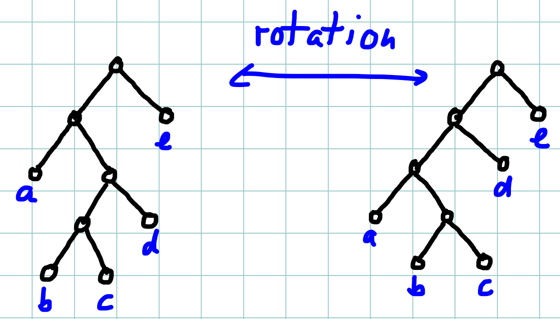
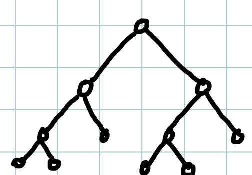
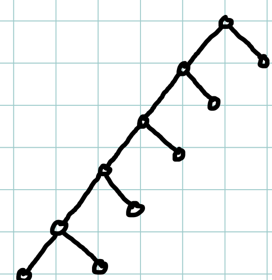


$(a\,((b\,c)\,d))\,e$ $\qquad$ $((a\,(b\,c))\,d\,)\,e$



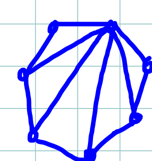
balanced

$O(\log n)$



$O(n)$ unbalanced

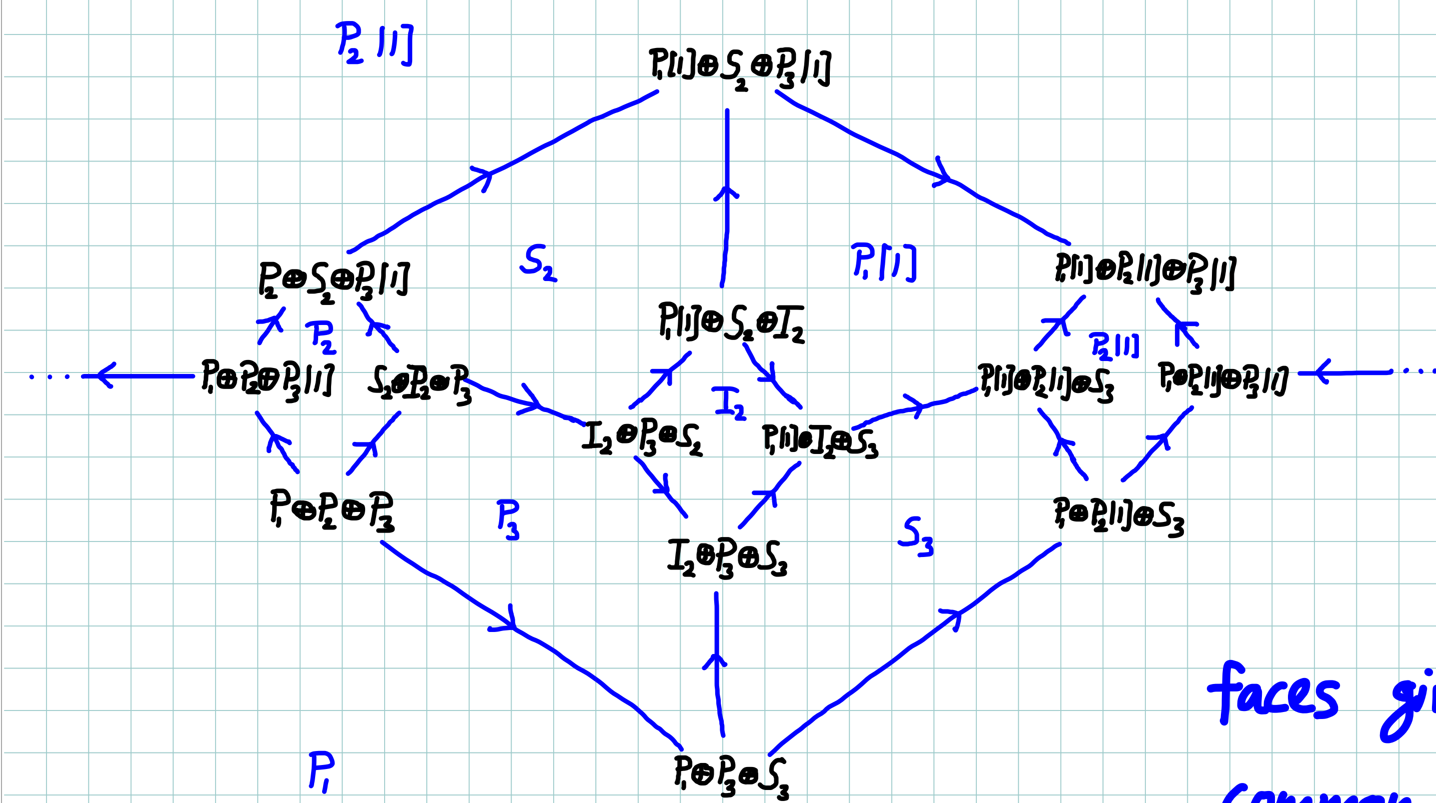
$\triangleq$ Definition 1



fan

The associahedron (type A_n)

Definition 3: vertices = cluster tilting objects type A_n
edges = mutations

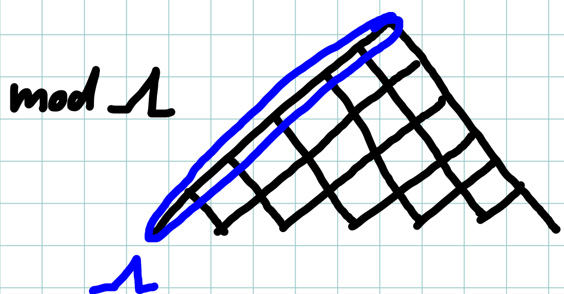


faces given by common summands

The associahedron type A_n

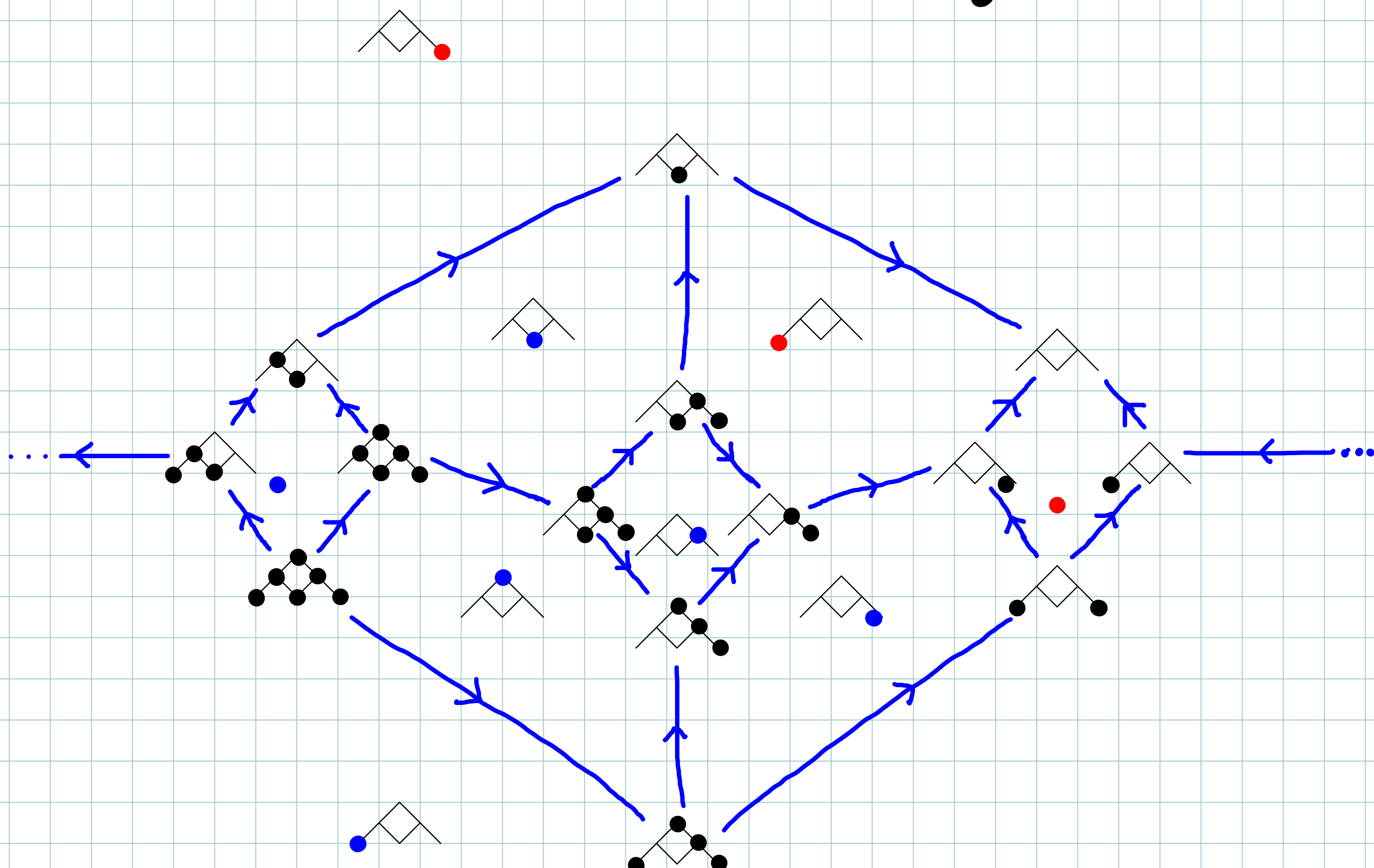
Definition n: vertices = tilting modules over $\Lambda = K(1 \to 2 \to \cdots \to n+1)$

edges = mutations (tilts)



The associahedron type $A_n = k(1 \to 2 \to \cdots \to n)$

Definition n+1: vertices = torsion classes in mod A_n
edges = edges of the Hasse quiver given by inclusion



Question: What is $\mathrm{diam}(A_n) = \max \mathrm{dist}(T, T')$?

[STT]

T, T' triangulation
tree
tilting object
torsion class

Observe:

n	1	2	3	4	5	6	7	8	9	10	11	12	13
$\mathrm{diam}(A_n)$	1	2	4	5	7	9	11	12	15	16	18	20	22

Note: $\mathrm{diam}(A_n) > \mathrm{dist}(B, B[1])$ for any $B = \mathrm{End}\, T$

max. green
sequences

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STT, 1988: $\mathrm{diam}(A_n) = 2n-4$ for $n \gg 0$

Pournin, 2014: $\mathrm{diam}(A_n) = 2n-4$ for $n \geq 10$

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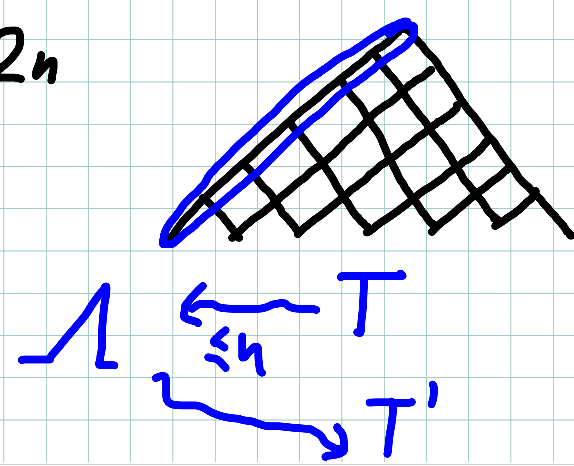
easy: $\mathrm{diam}(A_n) \leq 2n$



C. Ceballos, V. Pilaud, 2014:

NLF for types B, C (= cyclohedron), D, E_6, F_4, H_3, H_4.

Moreover, the diameter in type D_n is $2n-2$

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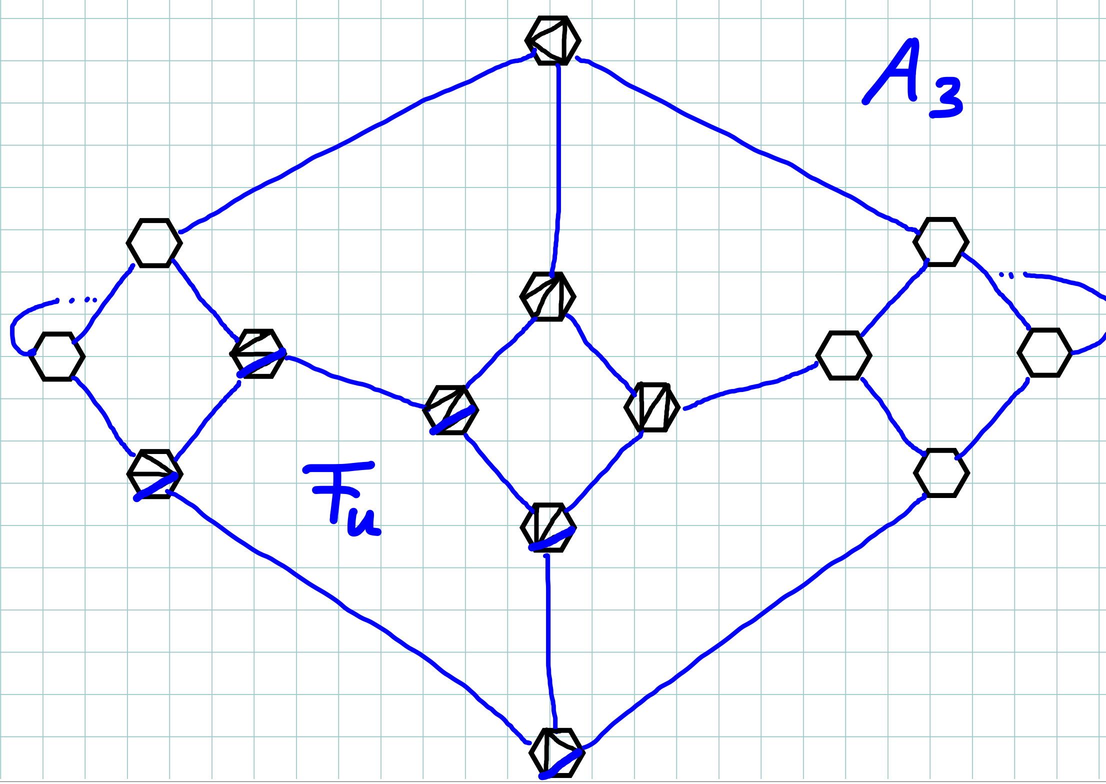
Y. Lebrun, J.-F. Marceau, 2014: 

NLF property is key ingredient to study diameter:

T, T' share $U \Leftrightarrow T, T' \in \mathcal{F}_U$ face of U

NLF: $\mathrm{dist}_{A_n}(T, T') = \mathrm{dist}_{\mathcal{F}_U}(T, T')$ if $T, T' \in \mathcal{F}_U$



<u>STT</u>: Key ingredient to prove NLF is a projection
$T \longmapsto p_u(T)$ onto the face $\mathcal{F}_u$ satisfying

(P1) $p_u(T) = T$ for $T \in \mathcal{F}_u$

(P2) $\mathcal{F}_u \ni T \underset{\text{edge}}{\text{———}} T' \quad \Rightarrow \quad p_u(T') = T$

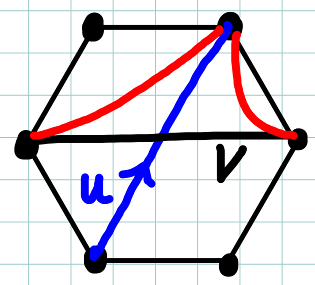
(P3) $\mathrm{dist}(T, T') \leq 1 \quad \Rightarrow \quad \mathrm{dist}(p_u(T), p_u(T')) \leq 1$

This proves NLF:

Given $u \in T — T' — T'' — \cdots — \tilde{T} \ni u$ apply p_u

$T — \qquad\qquad — \tilde{T}$ in $\mathcal{F}_u$

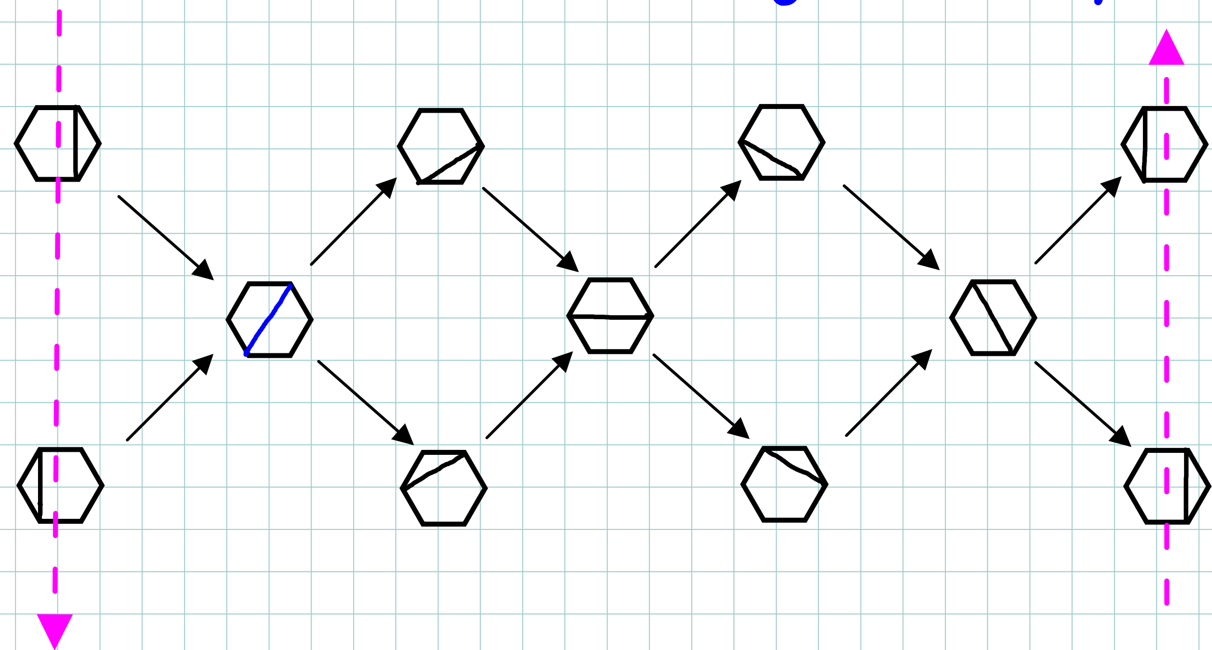
STT-projection:



- Choose an orientation on u
- For any $v \in T$, if it does not intersect u, keep it.
- If v intersects u, drag it along with u
- Add u if needed

Caldero, Chapoton, Schiffler, 2004:

Interpret diagonals of $(n+3)$-gon as indecomposable objects of $\mathcal{C}$ = cluster category of type A_n



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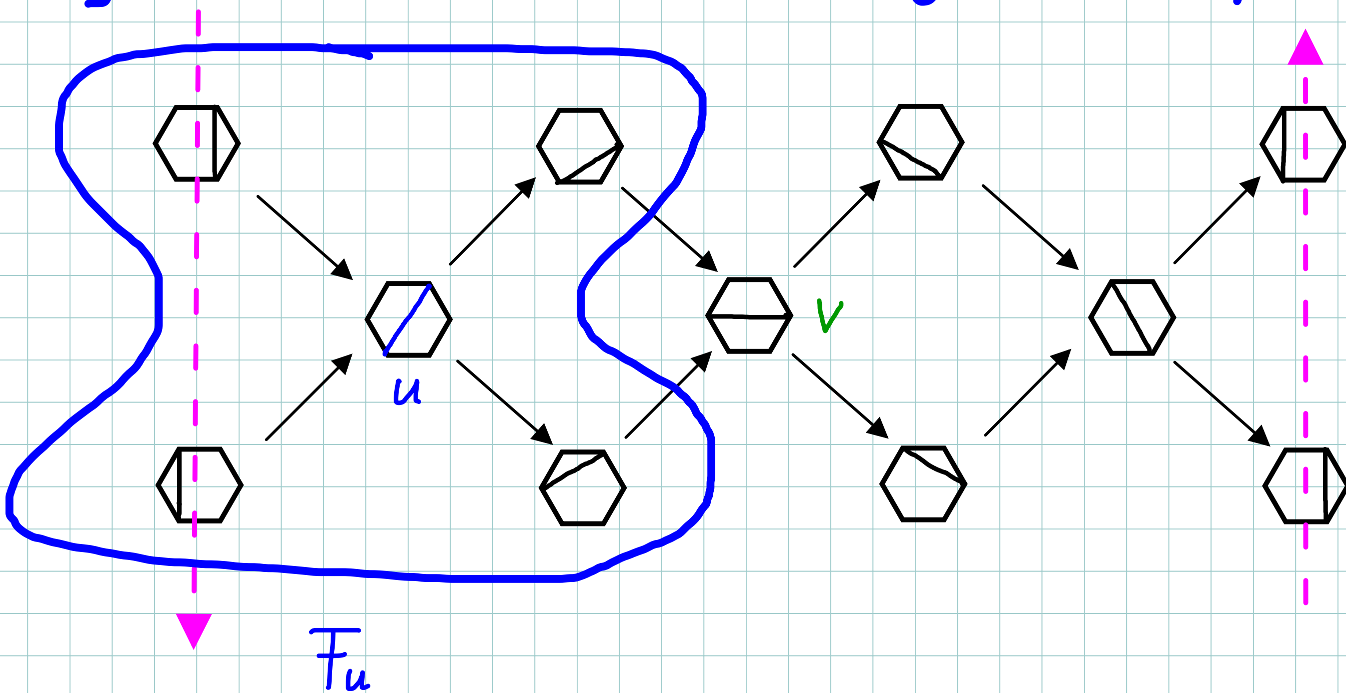

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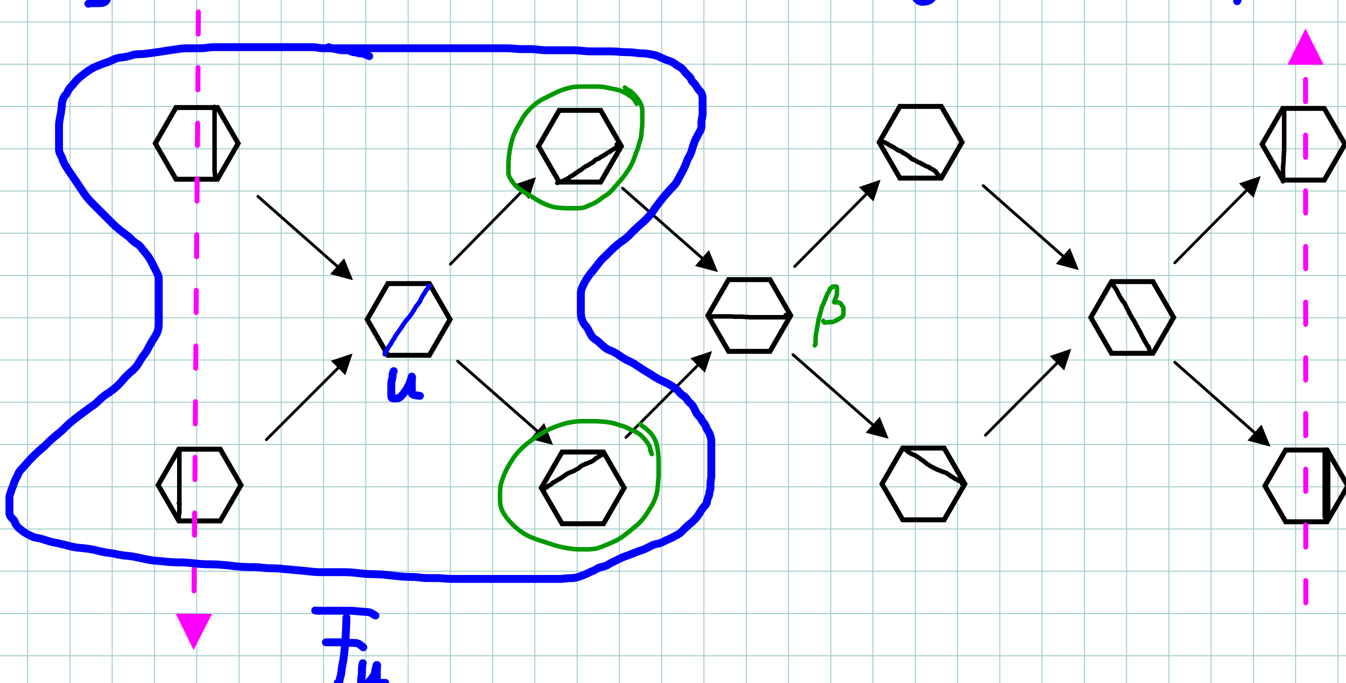

$\underbrace{\text{right } \mathcal{F}_u\text{-approximation of } V}_{RA_u(V)} \neq P_u(V)$

<u>The good news:</u> (B, Marceau 2014)

The right $\mathcal{F}_u$-approximation satisfies (P1), (P2), (P3)

(we verified this in types A_n and D_n, confirming NLF in these cases)

Generalizations:

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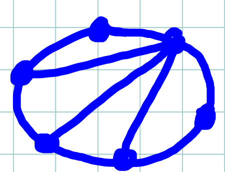
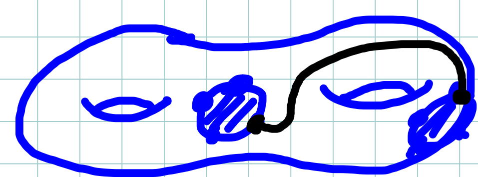


A_n



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(3) B, Treffinger, Wong, in progress:

$\mathcal{U}$ rigid object in a cluster category.

Then $\mathcal{F}_{\mathcal{U}} = {}^{\perp}(\mathcal{U}[1])$, $RA_{\mathcal{U}}(T) = B_{\mathcal{U}}(T)$

min. right approx. Bongartz completion (Jasso)

and (P1), (P2) holds