Wonder of sine-Gordon Y-systems
(joint with T. Nakanishi)

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Y-systems

- Systems of functional algebraic relations coming from the study of TBA.
- Actively studied in the '90 with ad-hoc methods.
- Usually complicated: it is hard to produce explicit solutions.
- Many of these system exhibit (in several cases conjectural) periodicity properties.

Classical Y-systems

Fix a finite type Dynkin diagram X. Let $A = (a_{mn})$ be the corresponding Cartan matrix.
Consider the family of commuting variables

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Definition

The *classical Y-system* associated to X is the system of algebraic relations

$$Y_m(u-1)Y_m(u+1) = \prod_{n \neq m} (1 + Y_n(u))^{-a_{mn}} \tag{1}$$

Zamolodchikov's Conjecture

Let h be the *Coxeter number* of X.

Conjecture

The solutions of (1) are periodic with period $2(h+2)$. That is, for any $m \in X$ and $u \in \mathbb{Z}$,

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Proofs

- When X is of type A_n the conjecture was proved independently by Frenkel-Szenes and Gliozzi-Tateo constructing the explicit solution.
- For general X the conjecture was proved by Fomin-Zelevinsky using y-pattern of cluster algebras.

Idea of the general proof

- Let $X = X_+ \sqcup X_-$ be a bipartition of X such that $m \in X_{\varepsilon(m)}$.
  Then the Y-system (1) only involves variables $\{Y_m(u)\}$ with a fixed parity of $\varepsilon(m)(-1)^u$.

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- Impose $Y_m(u) = Y_m(u+1)$ if $\varepsilon(m) = (-1)^u$ and combine with (1) to get

$$Y_m(u+1) = \begin{cases} \frac{\prod_{n \neq m}(1+Y_n(u))^{-a_{mn}}}{Y_m(u)} & \varepsilon(m) = (-1)^{u+1} \\ Y_m(u) & \varepsilon(m) = (-1)^u \end{cases} \tag{2}$$

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- Realize that (2) is the y-pattern evolution for a particular sequence of mutation (*bipartite*) in a cluster algebra of type X

General philosophy

Periodic behaviour in Y-systems and cluster algebras are intimately related:

to any sequence of mutations fixing a seed of a cluster algebra corresponds an (*explicit*) periodic Y-system.

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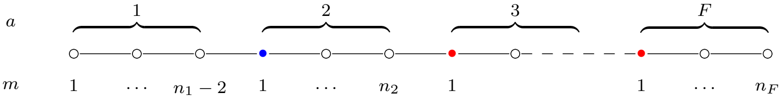
The same holds for any sequence of mutations fixing a seed up to relabeling.

Reduced sine-Gordon (RSG) and sine-Gordon (SG) Y-systems

- Generalization of classical Y-systems of types A and D respectively introduced by Tateo in 1995.
- Obtained by grouping the variables into blocks (*generations*) and prescribing different time evolutions for each block.
- The construction is "exotic": it involves continued fractions.
- The equations involved are complicated but, surprisingly, the conjectural periodicity is quite easy.

Reduced sine-Gordon Y-system

Let $X_{\mathrm{RSG}}(n_1, \ldots, n_F)$ be the Dynkin diagram of type A indexed by pairs (a, m) as follows:



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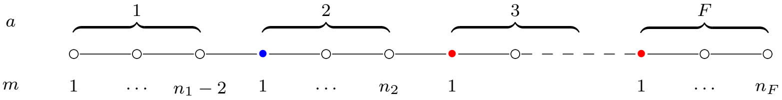

To $X_{\mathrm{RSG}}(n_1, \ldots, n_F)$ associate the continued fractions

$$\xi_a = [n_a, \ldots, n_1] := \cfrac{1}{n_a + \cfrac{1}{n_{a-1} + \cfrac{1}{\ddots + \cfrac{1}{n_1}}}}. \tag{3}$$

Write ξ_a as ratio of coprime integers:

$$\xi_a =: \frac{p_a}{q_a}$$

and set $r_a := p_a + q_a$.
Set also $\varepsilon_a := (-1)^{a-1}$.

- For a general (a, m) other than $(2,1)$, $(3,1)$, $\ldots$, $(F,1)$

$$Y_m^{(a)}(u-p_a)Y_m^{(a)}(u+p_a) = \prod_{(b,k)\sim(a,m)} (1+Y_k^{(b)}(u)^{\varepsilon_b})^{\varepsilon_b},$$

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- For $(a,m)=(2,1)$ (i.e. the blue vertex)

$$\begin{aligned}Y_1^{(2)}(u-p_2)Y_1^{(2)}(u+p_2)=&\,(1+Y_2^{(2)}(u)^{-1})^{-1}(1+Y_1^{(1)}(u))\\&\times\prod_{m=1}^{n_1-2}(1+Y_m^{(1)}(u-1-m)^{-1})^{-1}\\&\times\prod_{m=1}^{n_1-2}(1+Y_m^{(1)}(u+1+m)^{-1})^{-1}.\end{aligned}$$

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- For $(a,m)=(a,1)$ with $a=3,\ldots,F$ (i.e. the red vertices)

$$\begin{aligned} & Y_1^{(a)}(u-p_a)Y_1^{(a)}(u+p_a) \\ & \quad = (1+Y_2^{(a)}(u)^{\varepsilon_a})^{\varepsilon_a}(1+Y_{n_{a-2}-2\delta_{a3}}^{(a-2)}(u)^{\varepsilon_a})^{\varepsilon_a} \\ & \qquad \times \prod_{m=1}^{n_{a-1}}(1+Y_m^{(a-1)}(u-p_a+(n_{a-1}+1-m)p_{a-1})^{\varepsilon_a})^{\varepsilon_a} \\ & \qquad \times \prod_{m=1}^{n_{a-1}}(1+Y_m^{(a-1)}(u+p_a-(n_{a-1}+1-m)p_{a-1})^{\varepsilon_a})^{\varepsilon_a}, \end{aligned}$$

Tateo's conjecture (for RSG Y-systems)

Conjecture

The reduced Sine-Gordon Y-system associated to $X_{\mathrm{RSG}}(n_1, \ldots, n_F)$ is periodic with period $2r_F$. That is

$$Y_m^{(a)}(u + 2r_F) = Y_m^{(a)}(u)$$

for any $(a, m) \in X_{\mathrm{RSG}}(n_1, \ldots, n_F)$ and any $u \in \mathbb{Z}$.

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Moreover if L denotes the Rogers dilogarithm then

$$\frac{6}{\pi^2} \sum_{\substack{(a,m,u) \in \mathcal{I}_+ \\ 0 \leq u < 2r_F}} L\left(\frac{1}{1 + Y_m^{(a)}(u)}\right) = r_F \left(\sum_{a:\text{even}} n_a + 2\right) - 6r_F^{(2)}$$

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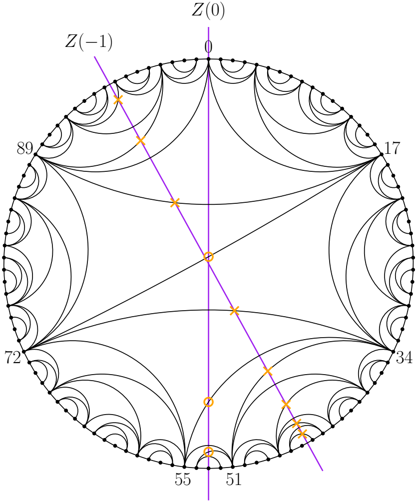
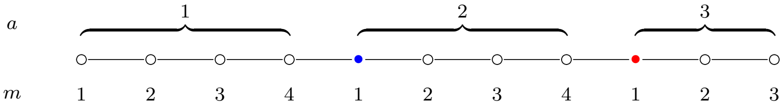
Theorem [Nakanishi,-]

Tateo's conjecture holds

Example: $X_{\mathrm{RSG}}(6,4,3)$





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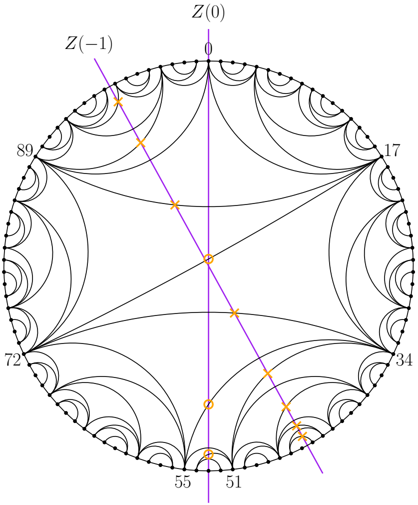
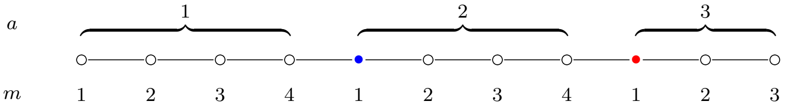


This triangulation represents a seed in a cluster algebra $\mathcal{A}$ of type A_{103}.

(It is a 106-gon)



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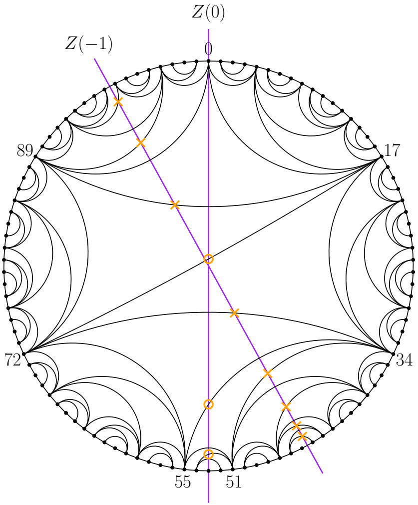
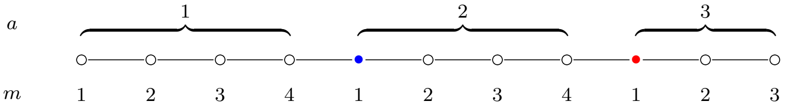


The conjectured periodicity of this Y-system is 212; indeed

$$\xi_3 = \frac{1}{3+\frac{1}{4+\frac{1}{6}}} = \frac{81}{25}$$

and

$$r_3 = 81 + 25 = 106$$



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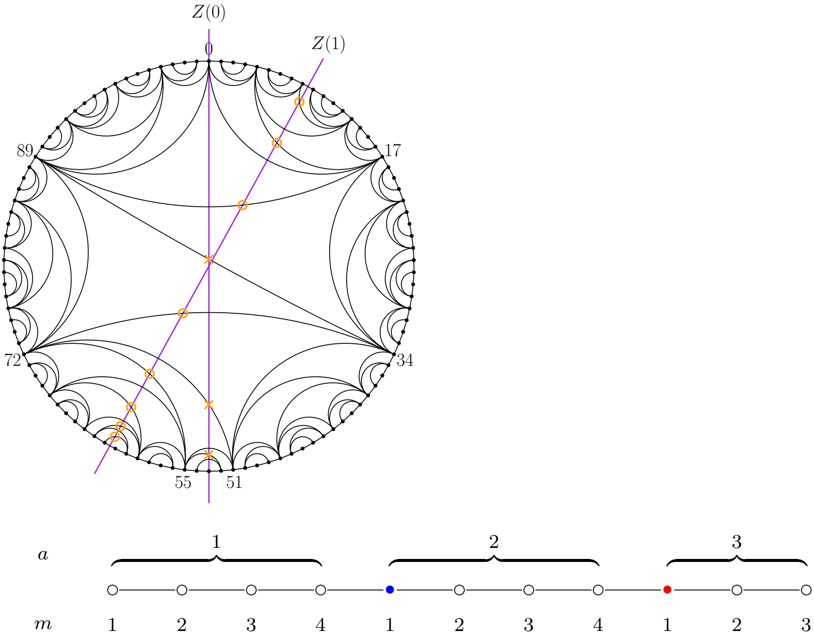

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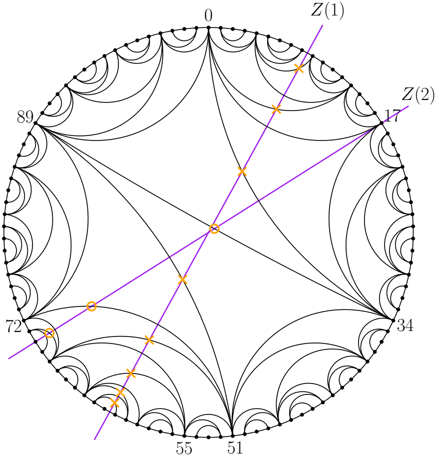
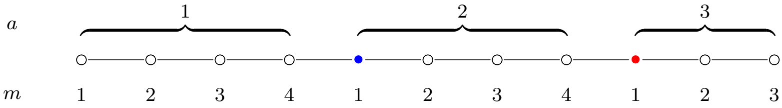


By reflecting (mutating) twice along different axes we rotated the picture by 17 steps.

But 17 and 106 are coprime so we need to reflect 212 times to go back to the original triangulation.



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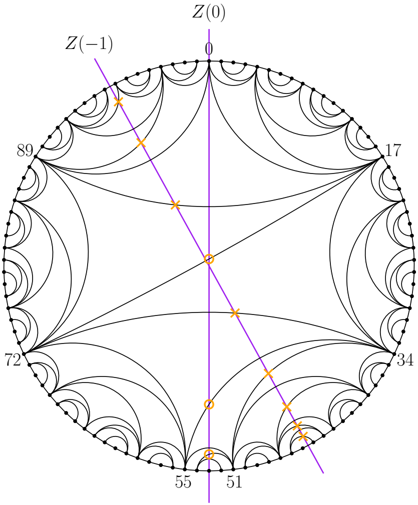
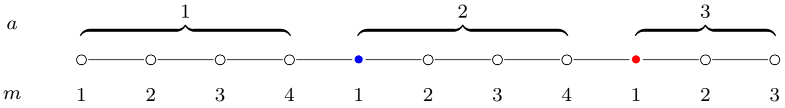


We need to perform some identifications to associate the variables in our Y-system to coefficients in the cluster algebra $\mathcal{A}$.

The triangulation contains precisely $4+4+3$ "different" arcs.



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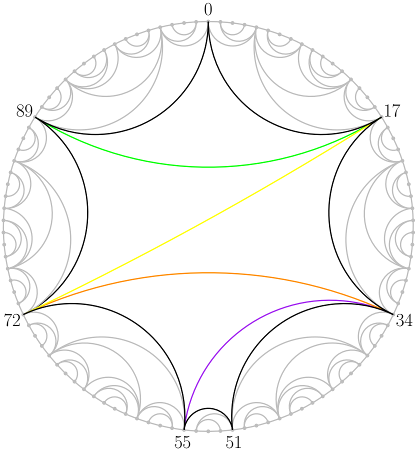
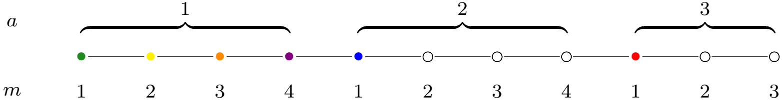


$p_1 = 1$ copy of the first generation.



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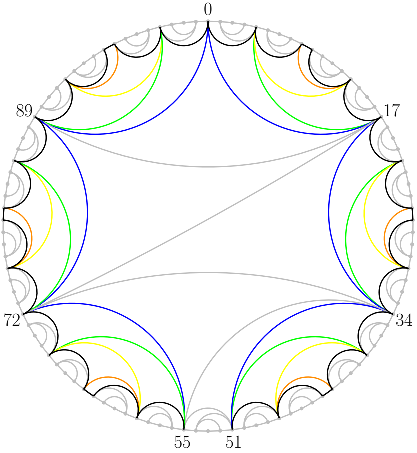
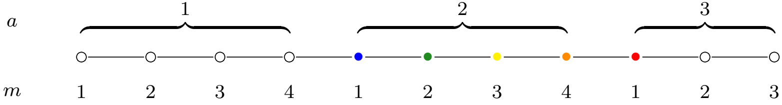


$p_2 = 6$ copies of the second generation.



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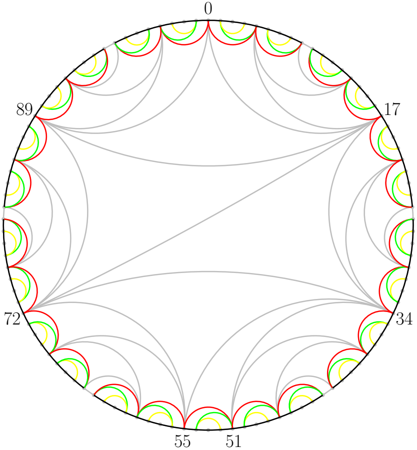

$p_3 = 25$ copies of the third generation.