Cluster algebras & Integrable systems

1) Cluster algebra

no definition

2) Poisson structure ~~tool~~ making functions into vector field (dynamical ~~systems~~ flows on M^n)

M^n – n dim mnfd

$$C^\infty(M^n) \xrightarrow{\text{Poisson str}} Vect(M^n)$$

Vector fields form Lie algebra

⇒ Functions ~~have to~~ must also form Lie algebra w.r.t. ~~{ }~~ Poisson bracket: { }

$$C^\infty(M) \otimes C^\infty(M) \longrightarrow C^\infty(M) \text{ satisfying}$$

- skew-symmetric $\{f, g\} = -\{g, f\}$
- Leibnitz rule (differentiation) $\{fg, h\} = \{f, h\}g + \{g, h\}f$
- Jacobi identity (Lie alg): $\{\{f, g\}h\} + \circlearrowleft = 0$

M is a Poisson mnfd

HAMILTONIAN SYSTEM: H – hamiltonian (function on M)

x_i – local coord. on M. $\dot{x}_i = \{x_i, H\}$ – diff. eqns / dynamical system

We want to consider cluster functions as regular functions on algebraic Poisson variety.

Def. Poisson ~~struct~~ bracket is compatible with collection of functions $\{f_1 .. f_n\}$ if $\{f_i, f_j\} = c_{ij} f_i f_j$ (or $\{\log f_i, \log f_j\} = c_{ij}$ (const))

Def. Poisson ~~str~~ bracket is compatible with cluster algebra if it is compatible with ~~any~~ every cluster:

$$\{f_i^t, f_j^t\} = c_{ij}^t f_i^t f_j^t \quad (c_{ij}^t \text{ depend on } t)$$

Remark. Jacobi id. is satisfied automatically.

$\Rightarrow$ any skew-sym. collection of c_{ij}^t determines Poisson bracket, by Leibnitz id.

Hami Cl

Ex A is given by cluster $(x_1, x_2; x_3)$ ← frozen

$$B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \\ \hline 0 & 2 \end{pmatrix}$$

$$\{x_1, x_2\} = \alpha x_1 x_2$$
$$\{x_1 x_3\} = \beta x_1 x_3$$
$$\{x_2 x_3\} = \gamma x_2 x_3$$

Mutation in direction ② gives:

$$x_2' = \frac{x_1 + x_3^2}{x_2}$$

$$\Rightarrow \{x_1, x_2'\} = \left\{x_1, \frac{x_1 + x_3^2}{x_2}\right\} = \frac{2x_3}{x_2}\{x_1 x_3\} -$$

$$- \frac{(x_1 + x_3^2)}{x_2^2}\{x_1 x_2\} = -2\beta \frac{x_1 x_3^2}{x_2} - \frac{2x_1(x_1 + x_3^2)}{x_2}$$

Can be written as product x_1 & $\frac{x_1 + x_3^2}{x_2}$ iff $\beta = 0$.

Similarly, considering $\{x_2', x_3\}$ we get $\gamma = 0$.

α is arbitrary

Q: Does compatible Poisson str. always exist?

Ex (rank $B < n$)

$\Rightarrow \alpha = \beta = \gamma = 0$

Yes if rank B is maximal.

§ y-coordinates: ~~Ex~~

$B_{(n+m)\times n}$, Take $\hat{B}_{(n+m)\times(n+m)}$ - skew-symmetric such that $\hat{B}_{(n+m)\times(n)} = B$.

Fix $\bar{x} = (x_1 \ldots 0, x_{n+1} \ldots x_{n+m})$ ~~$x_1 \ldots x_m$~~ - integers

set $y_i = x_i^{x_i} \prod_{k=1}^{n+m} x_k^{\hat{B}_{ki}}$,

~~Lemma (y~~ cluster y-dynamics

cluster mutations in direction k: $y_j' = \begin{cases} \frac{1}{y_k} & \text{if } j = k \\ y_j (1 + y_k)^{-B_{kj}} & \text{if } B_{kj} < 0 \\ y_j (1 + 1/y_k)^{-B_{kj}} & \text{if } B_{kj} > 0 \end{cases}$

<u>Def.</u> A square matrix A is <u>reducible</u> if A is transformed to a block-diag. form by permutations.

$r(A)$ = # of blocks.

Partition into blocks defines an equivalence relation on the rows (columns) of A.

<u>Thm</u>. Assume $B_{n+m,n}$ is skew-symmetr. and rank $B = n$

Then compatible Poisson brackets form a vector space of dim $r(B_{nn}) + \binom{m}{2}$

W.r.to basis y the Poisson bracket ~~matrix~~ to

$$\{ y_i, y_j \} = \omega_{ij} \, y_i y_j$$

with $[W] = \Lambda \hat{B} \hat{D}^{-1}$

Λ: $\mathrm{diag}\{\lambda_i\}$, $\lambda_i = \lambda_j$ if $i \sim j$

$\hat{B}$: extended exchange matrix

$\hat{D}$: diagonalizing matrix

skew-symmetric

Cor: if B is ~~square~~ irreducible of max. rank then

$$[W]_{(n+m) \times n} = \lambda B \hat{D}^{-1}$$

Rmrk. Compatible Poisson bracket is defined in y coord. by coeff. matrix. Com. quiver contains mutation quiver

Conj. Fomin Zel. "cluster determines seed"
"cluster variables determine exchange matrix"

Pf. ~~roughly~~ if rank $B=n$. W determines B

then, if W depends on $\{f_i\}$ only.

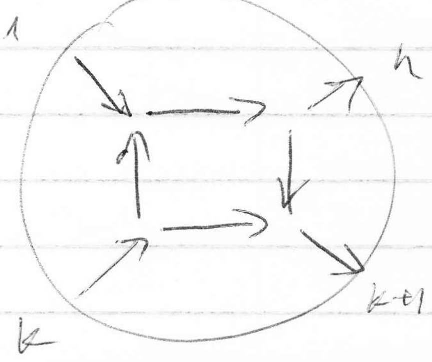
$\Rightarrow$ if $\{f_i\}$ return back $\Rightarrow$ I returns back $\Rightarrow$

$\Rightarrow$ B returns back.

Proved [Buan, Marsh, Reineke, Reiten, Todorov]
introducing cluster category.

Planar networks and Poisson structures

Planar network



3 valent vertices inside no sources, no sinks
univalent on the bndry.

edges equipped with ~~positive~~ numerical weights $w(e)$

k - sources on the bndry
n - bndry vertices.

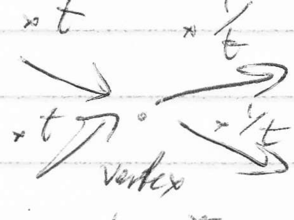
- Parametrizes part of grassmannian $G_k(n)$

Bndry measurement $B_{\text{source } i;\ \text{sink } j} = \sum_{\text{path}_p:\ i \to j} (-1)^{\text{wt}(P)} \prod_{e \in P} w(e)$

Planar network ⇝ {collection ($k \times (n-k)$ matrix) of boundary measurements B_{ij}.

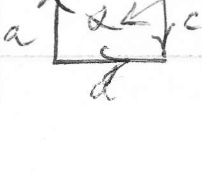
Equivalences not changing bndry measurements:

Gauge group.



$t \in \mathbb{C}^*$

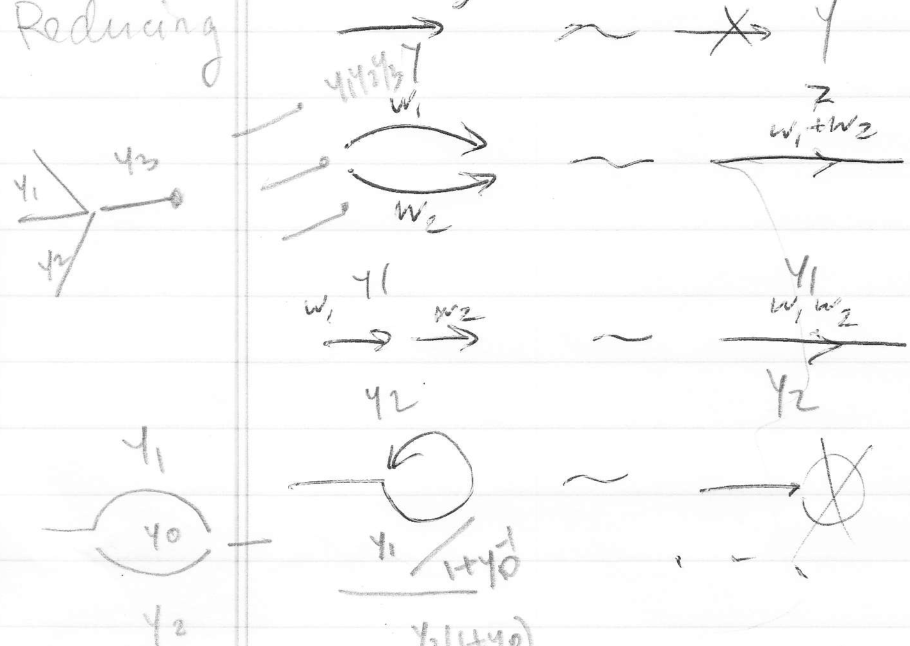
$\mathbb{C}^{*\,|E_\Gamma|} \big/ \text{gauge group} = \{\text{face coordinates}\}$ parametrize bndry measurements



$y_\alpha = a^{-1} b^{-1} c^{-1} d$

Elementary transformations

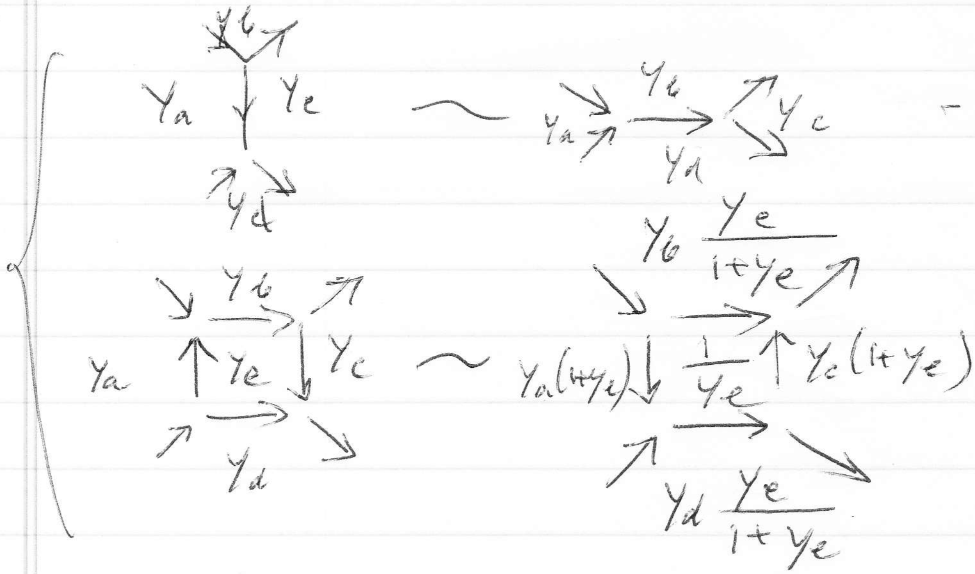
Reducing ~~Decreasing~~:



Postnikov thm.

~~One~~ using combin. of all moves one can reduce network to a minimal form

~~Nondecreasing~~: Nonreducing



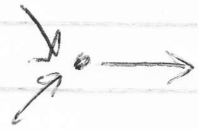
– id.

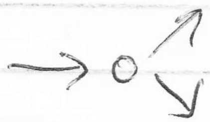
– Y-type cluster transformation.

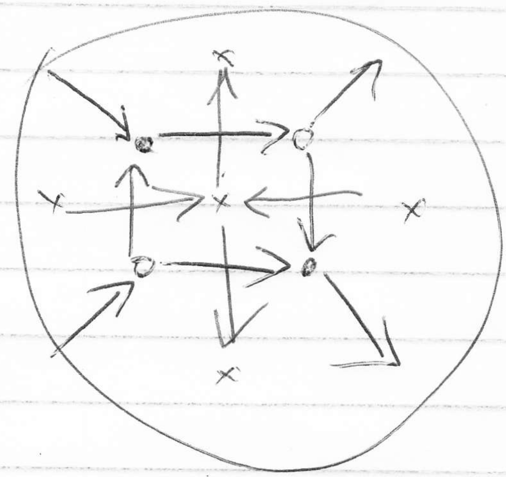
several such forms are related by similar transf. only

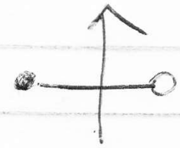
Do not change bndry measurements.

Cluster alg ⇝ Poisson str bracket
quiver = "dual graph to network"

black vertex 

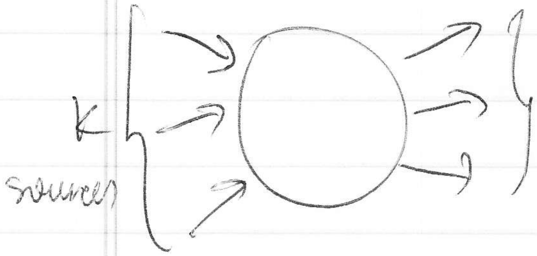
white vertex 



dual: 

Dual graph corresponds to matrix of compatible Poisson structure in face coordinates (gives exchange matrix)

This Poisson structure has nice properties (Poisson - Lie).



→ Boundary measurements form $k \times k$ matrix.

$$\mathrm{Mat}_k, \{\ ,\ \} \to \mathrm{Mat}_k \times \mathrm{Mat}_k, \{\ \}_{\mathrm{Mat}^2}$$

<u>Def</u>. $A \xrightarrow{T} B$ is Poisson if

$\{\ \}_A \qquad \{\ \}_B$, f, g

$$\{T^* f, T^* g\} = T^*\{f, g\}$$

mult: $\mathrm{Mat}_k \times \mathrm{Mat}_k \xrightarrow{m} \mathrm{Mat}_k$

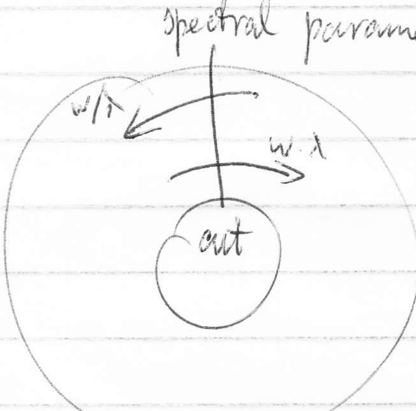
$\{\ \}_{\mathrm{Mat}_k^2} \qquad \{\ \}_{\mathrm{Mat}_k}$

~~<u>Def</u>. $\{\ \}_{\mathrm{Mat}_k}$ is Poisson~~

G – Lie group. $\{\ \}$ is Poisson Lie if $G \times G \xrightarrow{\text{mult}} G$ is a Poisson map.

<u>Thm</u> Poisson structure on face coordinates induces (6 parameter family of) (R-matrix) Poisson-Lie structure on Mat_k.

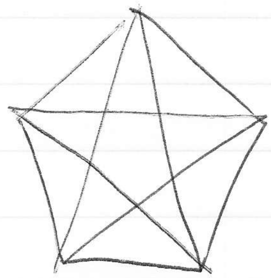
Network on an annulus.



Boundary measurement is a rat. function of λ.

- ∃ associated cluster algebra and compatible Poisson-Lie structure on ~~Rat~~ $Mat_k(\lambda)$ (a trigonometric R matrix)

Pentagram map



T_n : proj. classes ~~of proj~~ n-gons in $\mathbb{P}^2$ ⟲

~~twisted~~

<u>Def</u>. Twisted n-gon $\{ V_i \mid V_i \in \mathbb{P}^2 \}$

$V_{i+n} = M(V_i)$, where M is a fixed monodromy projective transformation on $\mathbb{P}^2$.

$\mathcal{P}_n$ = space of proj. classes twisted n-gons

Pentagram map $T_n : \mathcal{P}_n$ ⟲ (studied by R. Schwarz)

$\dim \mathcal{P}_n = 2n$.

<u>Main result</u> (Tabachnikov, Ovsienko, Schwarz) T_n is completely integrable / has maximal possible number of conservation laws / (integrals)

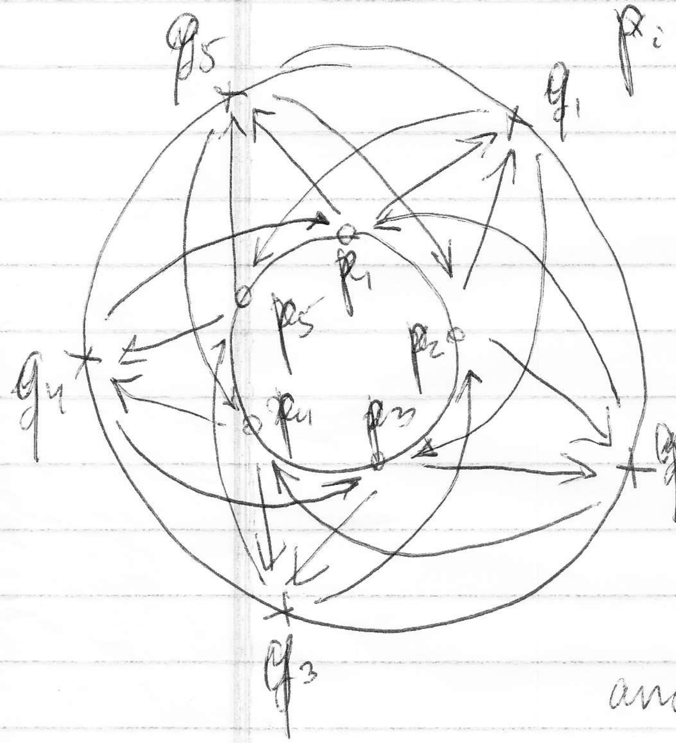
More exactly, $\exists$ an T_n-invariant Poisson structure ω on $\mathcal{P}_n$

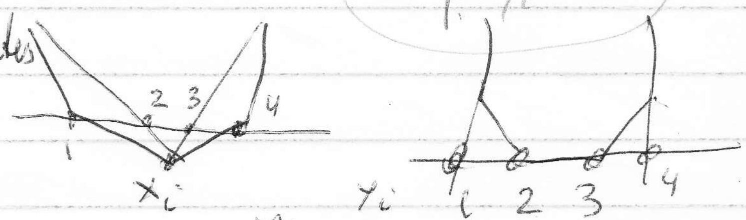
corank $\omega = \begin{cases} 2 & n \text{ is odd} \\ 4 & n \text{ is even} \end{cases}$ There are $\begin{cases} n+1 & \text{for odd } n \\ n+3 & \text{for even } n \end{cases}$ invariant functions I_j on $\mathcal{P}_n$ in involution such that $\omega\{I_i, I_j\} = 0$

Idea of the proof M. Glick "not quite" coordinates

$\prod p_i q_i = 1$

Gli I $n=5$ $2n$-coordinates p_i, q_i



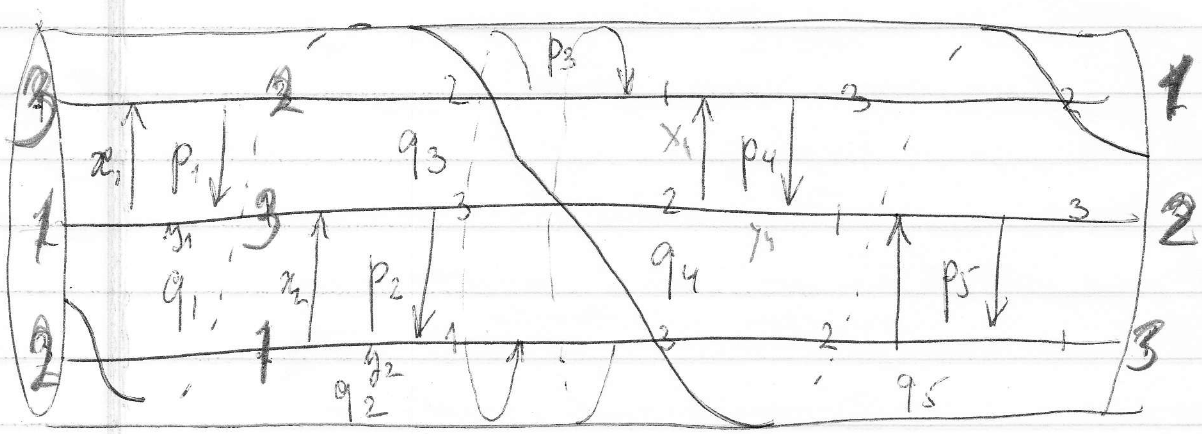


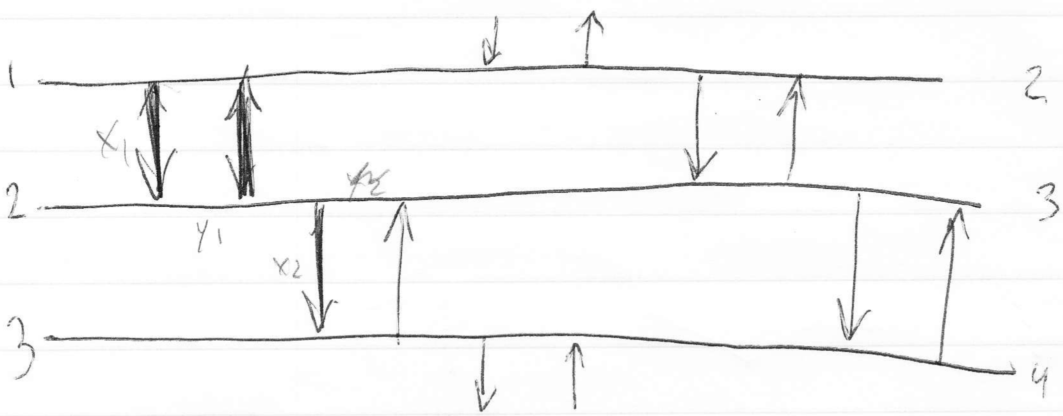
~~T_n = composition of clus~~

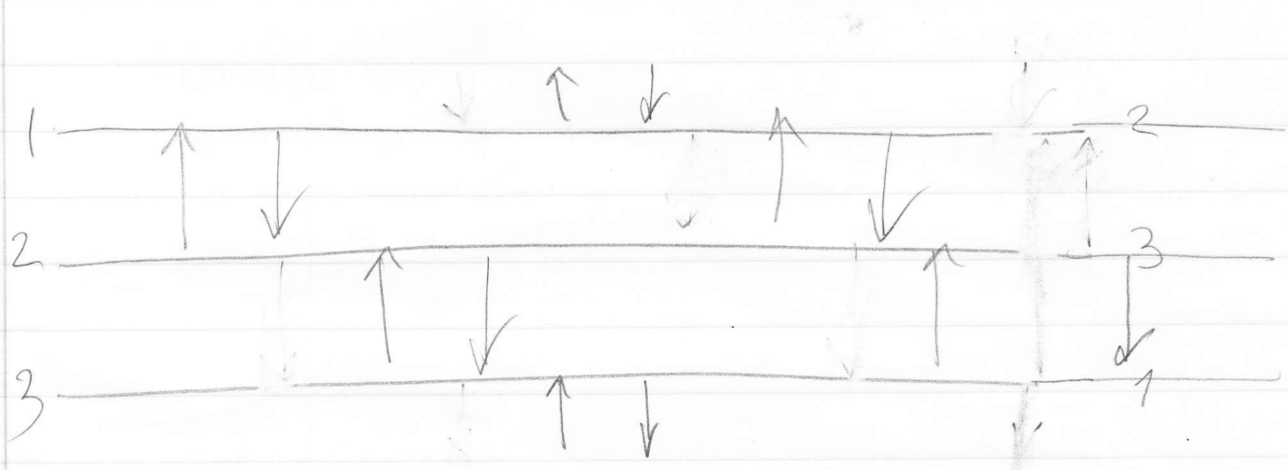
T_n written in $p_i\ q_i$ is a composition of y-type cluster transformations in all x_i and then replacing $x \leftrightarrow y$.

II Consider ~~the~~ network on a cylinder (annulus)

coordinates x_i, y_i

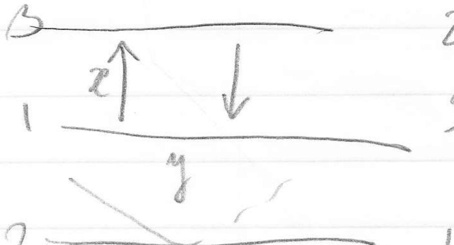






$p_i = \dfrac{y_i}{x_i}$

$q_i = \dfrac{x_{i+1}}{y_{i+1}}$



$$\sim \begin{pmatrix} 0 & x & x+y \\ \lambda & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

~~A~~$_i(\lambda)$

$M(\lambda) = \prod L_i(\lambda)$ in involution w.r.t. R-matrix Poisson str.

x characteristic polynomial $M(\lambda)$ is invariant.