Quasi-Cartan companions of cluster-tilted quivers

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April 2013

B: square matrix

B is *skew-symmetrizable* if

DA is *skew-symmetric* for some *diagonal* matrix D with positive diagonal entries.

- *Mutation* of B at an index k is the matrix $\mu_k(B) = B'$:

$$B' = \begin{cases} B'_{i,j} = -B_{i,j} & \text{if } i = k \text{ or } j = k; \\ B'_{i,j} = B_{i,j} + sgn(B_{i,k})[B_{i,k}B_{k,j}]_+ & \text{otherwise} \end{cases}$$

(where $[x]_+ = max\{x, 0\}$ and $sgn(x) = x/|x|$, $sgn(0) = 0$).

- *Mutation class* of B= all matrices that can be obtained from B by a sequence of mutations

B: skew-symmetrizable $n \times n$ matrix

Diagram of B is the directed graph such that

- vertices: $1, ..., n$
- $i \longrightarrow j$ if and only if $B_{j,i} > 0$
  - the edge is assigned the weight $|B_{i,j} B_{j,i}|$
  - (if the weight is 1 then we omit it in the picture)

Quiver notation:

Diagram of a skew-symmetric matrix = Quiver

- $B_{j,i} > 0$ many arrows from i to j

quiver notation

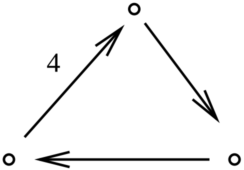


diagram notation

A: square matrix

A is *symmetrizable* if

> DA is *symmetric* for some *diagonal* matrix D with positive diagonal entries.

- A is called *positive* if C is positive definite
- A is called *semipositive* if C is positive semidefinite
- A is called *indefinite* if else.

B: skew-symmetrizable

A *quasi-Cartan companion* of B is a symmetrizable matrix A:

- $A_{i,i} = 2$
- $A_{i,j} = \pm B_{i,j}$ for all $i \neq j$.

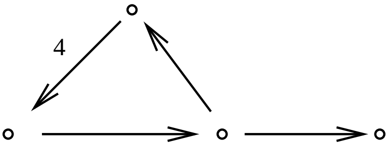
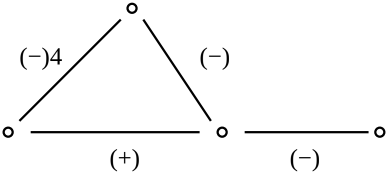


diagram of B



a quasi-Cartan companion of B

B: skew-symmetrizable

- B is called *finite (cluster) type* if for *any* B' which is mutation-equivalent to B, we have $|B'_{i,j}B'_{j,i}| \leq 3$ for all i, j.

Theorem (Barot-Geiss-Zelevinsky)

B is of finite type if and only if B has a quasi-Cartan companion A which is positive

Proof: "extend" mutation of B to a quasi-Cartan companion A

$$\mu_k(A) = A' = \begin{cases} A'_{k,k} = 2 & \\ A'_{i,k} = sgn(B_{i,k})A_{i,k} & \text{if } i \neq k \\ A'_{k,j} = -sgn(B_{k,j})A_{k,j} & \text{if } j \neq k \\ A'_{i,j} = A_{i,j} - sgn(A_{i,k}A_{k,j})[B_{i,k}B_{k,j}]_+ & \text{else} \end{cases}$$

- For B which is of infinite type, A' may not be a quasi-Cartan companion of $\mu_k(B)$

B: skew-symmetrizable

Definition: A companion of B is called *admissible* if

- each oriented cycle has an *odd* number of edges assigned $+$
- each non-oriented cycle has an *even* number of edges assigned $+$

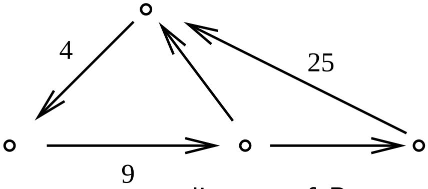
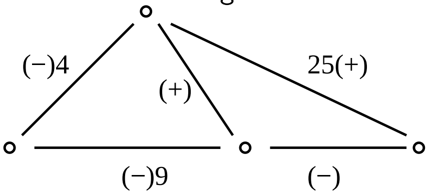


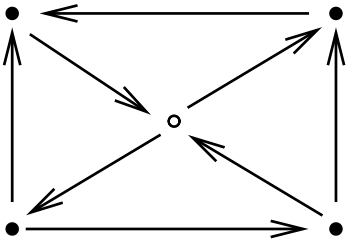
diagram of B



admissible companion

Theorem (S.) Any two admissible companions of B can be obtained from each other by a sequence of simultaneous sign changes in rows and columns.

However, an admissible companion may not exist!



- if $\Gamma(B)$ is acyclic, then B has an admissible companion: a generalized Cartan matrix ($A_{i,i} = 2$, $A_{i,j} = -|B_{i,j}|$)

B_0: skew-symmetric matrix such that $\Gamma(B_0)$ is acyclic
A_0: the generalized Cartan matrix associated to B_0

Theorem (S.) If B is mutation-equivalent to B_0, then B has an admissible quasi-Cartan companion A.

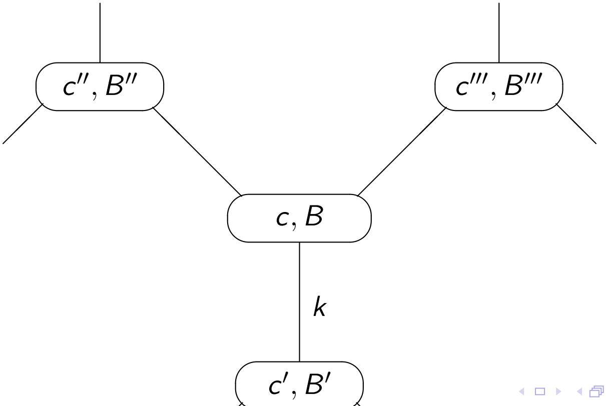
- A is obtained from A_0 by a sequence of mutations

In particular,

- if A is an admissible quasi-Cartan companion of B, then $\mu_k(A)$ is an admissible quasi-Cartan companion of $\mu_k(B)$

Proof: establish a particular companion, "**c**-vector companion"

$\mathbb{T}_n$: n-regular tree
t_0: initial vertex
$B_0 = B_{t_0}$: $n \times n$ skew-symmetrizable matrix (initial exchange matrix)
$\mathbf{c}_0 = \mathbf{c}_{t_0}$: standard basis of $\mathbb{Z}^n$
To each t in $\mathbb{T}_n$ assign $(\mathbf{c}_t, B_t) = (\mathbf{c}, B)$, a "$Y$-seed", such that $(\mathbf{c}', B') := \mu_k(\mathbf{c}, B)$:



- $B' = \mu_k(B)$
- The tuple $\mathbf{c}' = (\mathbf{c}'_1, \ldots, \mathbf{c}'_n)$ is given by

$$\mathbf{c}'_i = \begin{cases} -\mathbf{c}_i & \text{if } i = k; \\ \mathbf{c}_i + [sgn(\mathbf{c}_k)B_{k,i}]_+\mathbf{c}_k & \text{if } i \neq k. \end{cases} \tag{1}$$

Each $\mathbf{c}_i$ is sign-coherent: $\mathbf{c}_i > 0$ or $\mathbf{c}_i > 0$

(Derksen-Weyman-Zelevinsky, Demonet)

B: skew-symmetrizable $n \times n$ matrix such that $\Gamma(B)$ is *acyclic*
A: the associated generalized Cartan matrix
$\alpha_1, ..., \alpha_n$: simple roots
$Q = span(\alpha_1, ..., \alpha_n) \cong \mathbb{Z}^n$: root lattice
$s_i = s_{\alpha_i}: Q \to Q$: reflection

- $s_i(\alpha_j) = \alpha_j - A_{i,j}\alpha_i$

real roots: vectors obtained from the simple roots by a sequence of reflections

Theorem (Speyer, Thomas)

Each **c**-vector is the coordinate vector of a real root in the basis of simple roots.

B_0: *skew-symmetric* matrix such that $\Gamma(B_0)$ is acyclic
A_0: the associated generalized Cartan matrix
$(\mathbf{c}_0, B_0)$: initial Y-seed
$(\mathbf{c}, B)$: arbitrary Y-seed

Theorem (S.)

$A = (\mathbf{c}_i^T A_0 \mathbf{c}_j)$ is a quasi-Cartan companion of B

Furthermore:

- If $sgn(B_{j,i}) = sgn(\mathbf{c}_j)$, then $A_{j,i} = \mathbf{c}_j^T A_0 \mathbf{c}_i = -sgn(\mathbf{c}_j) B_{j,i}$.
- If $sgn(B_{j,i}) = -sgn(\mathbf{c}_j)$, then $A_{j,i} = \mathbf{c}_j^T A_0 \mathbf{c}_i = sgn(\mathbf{c}_i) B_{j,i}$.

In particular; if $sgn(\mathbf{c}_j) = -sgn(\mathbf{c}_i)$, then $B_{j,i} = sgn(\mathbf{c}_i)\mathbf{c}_j^T A_0 \mathbf{c}_i$.

More properties of the “**c**-vector companion” A :

- Every *directed path* of the diagram $\Gamma(B)$ has *at most one* edge $\{i, j\}$ such that $A_{i,j} > 0$.
- Every *oriented cycle* of the diagram $\Gamma(B)$ has *exactly one* edge $\{i, j\}$ such that $A_{i,j} > 0$.
- Every *non-oriented cycle* of the diagram $\Gamma(B)$ has an *even number* of edges $\{i, j\}$ such that $A_{i,j} > 0$.

B: skew-symmetric matrix

Definition

A set C of edges in $\Gamma(B)$ is called an "admissible cut" if

- every oriented cycle contains exactly one edge in C

  (for quivers with potentials, also introduced by Herschend, Iyama; for cluster tilting, introduced by Buan, Reiten, S.)
- every non-oriented cycle contains exactly an even number of edges in C.

If $\Gamma(B)$ is mutation-equivalent to an acyclic diagram, then it has an admissible cut of edges: those $\{i, j\}$ such that $A_{i,j} > 0$.

Equivalently:

if the diagram of a skew-symmetric matrix does not have an admissible cut of edges, then it is not mutation-equivalent to any acyclic diagram.