Two Perspectives on Cluster Mutations

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To get started defining the quantum cluster algebra we need the combinatorial data of a compatible pair $(\tilde{B}, \Lambda)$.

- $\tilde{B}$ - $m \times n$ $(m \geq n)$ exchange matrix
- B - skew-symmetrizable principal $n \times n$ submatrix
- D - diagonal skew-symmetrizing matrix, i.e. DB is skew-symmetric
- Λ - $m \times m$ commutation matrix

Compatibility Condition:

$$\tilde{B}^t \Lambda = \begin{pmatrix} D & 0 \end{pmatrix}$$

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For a parameter q, the commutation matrix Λ determines the quasi-commutation of an m-dimensional quantum torus $\mathcal{T}_{\Lambda,q}$ which will contain the quantum cluster algebra $\mathcal{A}_q(\tilde{B},\Lambda)$.

Quantum Torus:

$$\mathcal{T}_{\Lambda,q} = \mathbb{Z}[q^{\pm\frac{1}{2}}]\langle X_1^{\pm 1},\ldots,X_m^{\pm 1} : X_iX_j = q^{\lambda_{ij}}X_jX_i\rangle$$

The quantum torus has a unique anti-involution (reverses the order of products) called the bar-involution which fixes the generators ($\overline{X_i} = X_i$) and sends q to q^{-1}.

Bar Invariant Monomials ($\overline{X^{\mathbf{a}}} = X^{\mathbf{a}}$):

Let $\alpha_1,\ldots,\alpha_m$ be the standard basis vectors of $\mathbb{Z}^m$. For $\mathbf{a} = \sum_{i=1}^{m} a_i\alpha_i \in \mathbb{Z}^m$ we define bar-invariant monomials $X^{\mathbf{a}} = q^{-\frac{1}{2}\sum_{i<j} a_ia_j\lambda_{ij}} X_1^{a_1}\cdots X_m^{a_m}$.

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Write $\mathbf{X} = \{X_1, \ldots, X_m\}$ for the set of generators of the quantum torus $\mathcal{T}_{\Lambda,q}$ and call the collection $\mathbf{X}$ the initial cluster.

Initial Quantum Seed:

$$\Sigma_0 = (\mathbf{X}, \tilde{B}, \Lambda)$$

Let $\mathbb{T}_n$ denote the rooted n-regular tree with root vertex t_0. We will label the n edges of $\mathbb{T}_n$ emanating from each vertex by the set $\{1, \ldots, n\}$.

We will actually have many quantum seeds Σ_t, one for each vertex t of $\mathbb{T}_n$, subject to the following conditions:

- The initial quantum seed is associated to the root: $\Sigma_{t_0} = \Sigma_0$.
- If there exists an edge of $\mathbb{T}_n$ labeled by k between vertices t and t', then the quantum seeds Σ_t and $\Sigma_{t'}$ are related by the mutation in direction k.

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To define the mutation of quantum seeds we need a little more notation.

Write $\mathbf{b}^k$ for the k^{th} column of $\tilde{B}$ thought of as an element of $\mathbb{Z}^m$. Let $\mathbf{b}_+^k = \sum_{b_{ik}>0} b_{ik}\alpha_i$ and $\mathbf{b}_-^k = \mathbf{b}_+^k - \mathbf{b}^k$.

Internal Mutations:

For $1 \leq k \leq n$, define the mutation $\mu_k\Sigma = (\mu_k\mathbf{X}, \mu_k\tilde{B}, \mu_k\Lambda)$ of a seed in direction k as follows:

- $\mu_k\mathbf{X} = \mathbf{X} \setminus \{X_k\} \cup \{X'_k\}$ where $X'_k = X^{\mathbf{b}_+^k - \alpha_k} + X^{\mathbf{b}_-^k - \alpha_k}$,
- $\mu_k\tilde{B} = E_k\tilde{B}F_k$ (Fomin-Zelevinsky),
- $\mu_k\Lambda = E_k\Lambda E_k^t$ (Berenstein-Zelevinsky).

Note: cluster variables obtained through iterated mutations will, a priori, be elements of the skew-field of fractions $\mathcal{F}$ of $\mathcal{T}_{\Lambda,q}$.

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We are finally ready to define the quantum cluster algebra.

Quantum Cluster Algebra:

Define the quantum cluster algebra $\mathcal{A}_q(\tilde{B}, \Lambda)$ to be the $\mathbb{Z}[q^{\pm\frac{1}{2}}]$-subalgebra of $\mathcal{F}$ generated by all cluster variables from all seeds Σ_t where t runs over the vertices of the mutation tree $\mathbb{T}_n$.

Theorem (Quantum Laurent Phenomenon: Berenstein, Zelevinsky)

For any seed $\Sigma_t = (\mathbf{X}_t, \tilde{B}_t, \Lambda_t)$, the quantum cluster algebra $\mathcal{A}_q(\tilde{B}, \Lambda)$ is a subalgebra of the quantum torus $\mathcal{T}_{\Lambda_t, q}$.

Laurent Problem:

Understand the initial cluster Laurent expansion of each cluster variable.

Our Goal: Solve this problem when the principal submatrix B is acyclic.

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The internal mutation μ_k in the definition of the quantum cluster algebra should be viewed as a recursive process inside the fixed skew-field $\mathcal{F}$.

There is another way to look at mutations, we view the mutation as a change of the initial cluster. We will call this type of mutation an external mutation.

To be more precise suppose t and t' are connected by an edge in $\mathbb{T}_n$ labeled by k. By the quantum Laurent phenomenon the quantum cluster algebra $\mathcal{A}_q(\tilde{B}, \Lambda)$ is contained in both $\mathcal{T}_{\Lambda_t,q} \subset \mathcal{F}_t$ and $\mathcal{T}_{\Lambda_{t'},q} \subset \mathcal{F}_{t'}$.

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External Mutations:

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With regards to the Laurent problem these two mutations have close connections to the representation theory of valued quivers (species).

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Our solution to the Laurent problem will involve combinatorial objects called valued quivers.

Valued Quivers:

- $Q = (Q_0, Q_1, s, t)$ - acyclic quiver with vertices $Q_0 = \{1, \ldots, n\}$, arrows Q_1, and source and target maps $s, t : Q_1 \to Q_0$.
- $\mathbf{d} : Q_0 \to \mathbb{Z}_{>0}$ - valuations on the vertices, $\mathbf{d}(i) = d_i$.
- Call the pair $(Q, \mathbf{d})$ an *acyclic valued quiver*.

From a skew-symmetrizable $n \times n$ matrix B we can construct a valued quiver $(Q, \mathbf{d})$ as follows:

- Q has vertices $\{1, \ldots, n\}$ with valuations
$$d_i = i^{th} \text{ diagonal entry of the symmetrizing matrix } D,$$
- whenever $b_{ij} > 0$, Q has $gcd(b_{ij}, -b_{ji})$ arrows $i \to j$.

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- $Q = (Q_0, Q_1, s, t)$ - acyclic quiver with vertices $Q_0 = \{1, \ldots, n\}$, arrows Q_1, and source and target maps $s, t : Q_1 \to Q_0$.
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From a skew-symmetrizable $n \times n$ matrix B we can construct a valued quiver $(Q, \mathbf{d})$ as follows:

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A representation $V = (\{V_i\}_{i\in Q_0}, \{\varphi_a\}_{a\in Q_1})$ of $(Q, \mathbf{d})$ consists of an $\mathbb{F}_{d_i}$-vector space V_i for each vertex i and an $\mathbb{F}_{\gcd(d_{s(a)},d_{t(a)})}$-linear map $\varphi_a : V_{s(a)} \to V_{t(a)}$ for each arrow a.

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We define the quantum cluster character $V \mapsto X_V \in \mathcal{T}_{\Lambda,q}$ by

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where $Gr_{\mathbf{e}}(V)$ denotes the Grassmannian of subrepresentations of V with isomorphism class $\mathbf{e}$.

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The quantum cluster character $V \mapsto X_V$ defines a bijection from indecomposable rigid representations V of $(Q, \mathbf{d})$ to non-initial quantum cluster variables of the quantum cluster algebra $\mathcal{A}_q(\tilde{B}, \Lambda) \subset \mathcal{T}_{\Lambda,q}$.

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For $V \in \mathrm{rep}_{\mathbb{F}}(Q, \mathbf{d})\langle k\rangle$ the external mutation at a sink or source vertex k can be computed via the reflection functor Σ_k by

$$\tilde{\mu}_k(X_V) = X_{\Sigma_k V}.$$

Lemma

For any vertex k, the cluster variable obtained from the initial cluster by mutating in direction k is given by the quantum cluster character X_{S_k}.

A cluster $(\mathbf{X}', \tilde{B}', \Lambda')$ is called almost acyclic if there exists a vertex k so that $(\mu_k\mathbf{X}', \mu_k\tilde{B}', \mu_k\Lambda')$ is acyclic.

Corollary

Any cluster variable of $\mathcal{A}_q(\tilde{B}, \Lambda)$ in an almost acyclic cluster is given by X_V for some representation V which can be obtained via reflection functors from a simple representation.

Open Question: What about non-sink/non-source mutations?

...Ask Daniel Labardini-Fragoso...there appear to be obstructions.

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Let $V \in \text{rep}_{\mathbb{F}}(Q, \mathbf{d})$.

- V is rigid if $Ext^1(V, V) = 0$.
- V is basic if each indecomposable summand appears with multiplicity one.
- The support of V is the set $\text{supp}(V) = \{i \in Q_0 : V_i \neq 0\}$.
- V is sincere if $\text{supp}(V) = Q_0$.

Local Tilting Representations:

We will call a representation $T \in \text{rep}_{\mathbb{F}}(Q, \mathbf{d})$ *local tilting* if T is basic, rigid, and the number of indecomposable summands is equal to the number of vertices in its support.

Important: the zero representation is local tilting.

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Main Idea (Hubery): local tilting representations are in bijection with seeds

To make this precise we will need to recall two classical theorems on tilting in hereditary categories:

Theorem (Happel, Ringel)

Suppose T is basic and rigid. Then T is a tilting representation if and only if T has as many non-isomorphic indecomposable summands as the number of simple representations.

This allows us to restrict a local tilting representation T to the full subquiver of $(Q, \mathbf{d})$ on the vertices $\mathrm{supp}(T)$ where it becomes a tilting representation.

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$T \in \text{rep}_{\mathbb{F}}(Q, \mathbf{d})$ is called almost complete tilting if it contains one less than the required number of indecomposable summands.

Theorem (Happel, Unger)

Let T be an almost complete tilting representation. If T is sincere, then there exist exactly two non-isomorphic complements to T, otherwise there is a unique complement.

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Mutation of Local Tilting Representations:

Define the mutation $\mu_k(T) = T'$ in direction k as follows:

1. If vertex $k \notin \text{supp}(T)$, then there exists a unique complement T'_k so that $T' = T'_k \oplus T$ is a local tilting representation containing k in its support.
2. If vertex $k \in \text{supp}(T)$, then write $\overline{T} = T/T_k$.
   1. If $\overline{T}$ is a local tilting representation, i.e. $k \notin \text{supp}(\overline{T})$, let $T' = \overline{T}$.
   2. Otherwise $\text{supp}(\overline{T}) = \text{supp}(T)$ and there exists a unique compliment $T'_k \not\cong T_k$ so that $T' = T'_k \oplus \overline{T}$ is a local tilting representation.

It follows from results of [BMRRT] that every local tilting representation can be obtained from the zero representation by a sequence of mutations.

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We assign a quantum seed $\Sigma_T = (\mathbf{X}_T, \tilde{B}_T, \Lambda_T)$ to each local tilting representation T as follows:

- $\mathbf{X}_T = (X'_1, \ldots, X'_m)$ is given by

$$X'_k = \begin{cases} X_k & \text{if } k \notin \mathrm{supp}(T); \\ X_{T_k} & \text{if } k \in \mathrm{supp}(T); \end{cases}$$

- The k^{th} column of the exchange matrix $\tilde{B}_T$ is defined homologically in terms of T and T_k^* (Hubery);
- Λ_T records the quasi-commutation of $\mathbf{X}_T$ (explicitly given by formulas involving the Euler-Ringel form and Λ).

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Suppose $\mu_k(T) = T'$. Then Σ_T and $\Sigma_{T'}$ are related by Berenstein-Zelevinsky quantum seed mutation in direction k.

Lemma

The quantum seed associated to the zero representation is exactly the initial quantum seed $(\mathbf{X}, \tilde{B}, \Lambda)$.

Corollary (R.)

The quantum cluster character $V \mapsto X_V$ defines a bijection from indecomposable rigid representations of $(Q, \mathbf{d})$ to non-initial quantum cluster variables of the quantum cluster algebra $\mathcal{A}_q(\tilde{B}, \Lambda)$.

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