# Two Perspectives on Cluster Mutations

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- B skew-symmetrizable principal  $n \times n$  submatrix
- D diagonal skew-symmetrizing matrix, i.e. DB is skew-symmetric
- $\Lambda$   $m \times m$  commutation matrix

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Quantum Torus:

$$\mathcal{T}_{\Lambda,q} = \mathbb{Z}[q^{\pm \frac{1}{2}}] \langle X_1^{\pm 1}, \dots, X_m^{\pm 1} : X_i X_j = q^{\lambda_{ij}} X_j X_i \rangle$$

The quantum torus has a unique anti-involution (reverses the order of products) called the bar-involution which fixes the generators  $(\overline{X_i} = X_i)$  and sends q to  $q^{-1}$ .

## Bar Invariant Monomials $(\overline{X^{a}} = X^{a})$ :

Let  $\alpha_1, \ldots, \alpha_m$  be the standard basis vectors of  $\mathbb{Z}^m$ . For  $\mathbf{a} = \sum_{i=1}^m a_i \alpha_i \in \mathbb{Z}^m$ 

we define bar-invariant monomials  $X^{a} = q^{-\frac{1}{2}\sum_{i < j} a_{i}a_{j}\lambda_{ij}} X_{1}^{a_{1}} \cdots X_{m}^{a_{m}}$ .

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Quantum Seeds and the Mutation Tree

Write  $\mathbf{X} = \{X_1, \dots, X_m\}$  for the set of generators of the quantum torus  $\mathcal{T}_{\Lambda,q}$  and call the collection  $\mathbf{X}$  the initial cluster.

Initial Quantum Seed:

$$\Sigma_0 = (\mathbf{X}, \tilde{B}, \Lambda)$$

Let  $\mathbb{T}_n$  denote the rooted *n*-regular tree with root vertex  $t_0$ . We will label the *n* edges of  $\mathbb{T}_n$  emanating from each vertex by the set  $\{1, \ldots, n\}$ .

We will actually have many quantum seeds  $\Sigma_t$ , one for each vertex t of  $\mathbb{T}_n$ , subject to the following conditions:

- The initial quantum seed is associated to the root:  $\Sigma_{t_0} = \Sigma_0$ .
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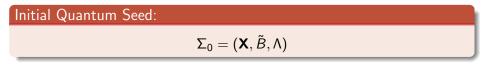
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## To define the mutation of quantum seeds we need a little more notation.

Write  $\mathbf{b}^k$  for the  $k^{th}$  column of  $\tilde{B}$  thought of as an element of  $\mathbb{Z}^m$ . Let  $\mathbf{b}^k_+ = \sum_{b_{ik}>0} b_{ik}\alpha_i$  and  $\mathbf{b}^k_- = \mathbf{b}^k_+ - \mathbf{b}^k$ .

### Internal Mutations:

For  $1 \le k \le n$ , define the mutation  $\mu_k \Sigma = (\mu_k \mathbf{X}, \mu_k \tilde{B}, \mu_k \Lambda)$  of a seed in direction k as follows:

• 
$$\mu_k \mathbf{X} = \mathbf{X} \setminus \{X_k\} \cup \{X'_k\}$$
 where  $X'_k = X^{\mathbf{b}^k_+ - \alpha_k} + X^{\mathbf{b}^k_- - \alpha_k}$ ,

- $\mu_k \tilde{B} = E_k \tilde{B} F_k$  (Fomin-Zelevinsky),
- $\mu_k \Lambda = E_k \Lambda E_k^t$  (Berenstein-Zelevinsky).

Note: cluster variables obtained through iterated mutations will, a priori, be elements of the skew-field of fractions  $\mathcal{F}$  of  $\mathcal{T}_{\Lambda,q}$ .



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Define the quantum cluster algebra  $\mathcal{A}_q(\tilde{B}, \Lambda)$  to be the  $\mathbb{Z}[q^{\pm \frac{1}{2}}]$ -subalgebra of  $\mathcal{F}$  generated by all cluster variables from all seeds  $\Sigma_t$  where t runs over the vertices of the mutation tree  $\mathbb{T}_n$ .

# Theorem (Quantum Laurent Phenomenon: Berenstein, Zelevinsky)

For any seed  $\Sigma_t = (\mathbf{X}_t, \tilde{B}_t, \Lambda_t)$ , the quantum cluster algebra  $\mathcal{A}_q(\tilde{B}, \Lambda)$  is a subalgebra of the quantum torus  $\mathcal{T}_{\Lambda_t,q}$ .

### Laurent Problem:

Understand the initial cluster Laurent expansion of each cluster variable.

# Our Goal: Solve this problem when the principal submatrix B is acyclic.

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There is another way to look at mutations, we view the mutation as a change of the initial cluster. We will call this type of mutation an external mutation.

To be more precise suppose t and t' are connected by an edge in  $\mathbb{T}_n$ labeled by k. By the quantum Laurent phenomenon the quantum cluster algebra  $\mathcal{A}_q(\tilde{B}, \Lambda)$  is contained in both  $\mathcal{T}_{\Lambda_t, q} \subset \mathcal{F}_t$  and  $\mathcal{T}_{\Lambda_{t'}, q} \subset \mathcal{F}_{t'}$ .

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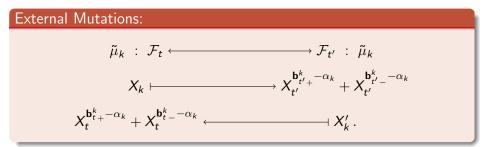
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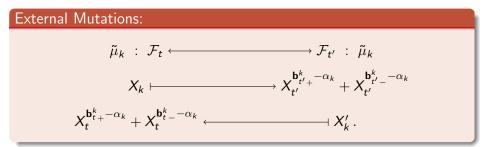
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Our solution to the Laurent problem will involve combinatorial objects called valued quivers.

### Valued Quivers:

- $Q = (Q_0, Q_1, s, t)$  acyclic quiver with vertices  $Q_0 = \{1, ..., n\}$ , arrows  $Q_1$ , and source and target maps  $s, t : Q_1 \to Q_0$ .
- $\mathbf{d}: Q_0 \to \mathbb{Z}_{>0}$  valuations on the vertices,  $\mathbf{d}(i) = d_i$ .
- Call the pair (Q, d) an acyclic valued quiver.

From a skew-symmetrizable  $n \times n$  matrix B we can construct a valued quiver  $(Q, \mathbf{d})$  as follows:

• Q has vertices  $\{1, \ldots, n\}$  with valuations  $d_i = i^{th}$  diagonal entry of the symmetrizing matrix D,

• whenever  $b_{ij} > 0$ , Q has  $gcd(b_{ij}, -b_{ji})$  arrows  $i \rightarrow j$ .

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Our solution to the Laurent problem will involve combinatorial objects called valued quivers.

## Valued Quivers:

- $Q = (Q_0, Q_1, s, t)$  acyclic quiver with vertices  $Q_0 = \{1, \ldots, n\}$ , arrows  $Q_1$ , and source and target maps  $s, t : Q_1 \to Q_0$ .
- $\mathbf{d}: Q_0 \to \mathbb{Z}_{>0}$  valuations on the vertices,  $\mathbf{d}(i) = d_i$ .
- Call the pair  $(Q, \mathbf{d})$  an *acyclic valued quiver*.

From a skew-symmetrizable  $n \times n$  matrix B we can construct a valued quiver  $(Q, \mathbf{d})$  as follows:

• Q has vertices  $\{1, \ldots, n\}$  with valuations  $d_i = i^{th}$  diagonal entry of the symmetrizing matrix D,

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- $\bar{\mathbb{F}}$  an algebraic closure of  $\mathbb{F}$
- $\mathbb{F}_k$  degree k extension of  $\mathbb{F}$  in  $\overline{\mathbb{F}}$
- Note:  $\mathbb{F}_k \cap \mathbb{F}_\ell = \mathbb{F}_{gcd(k,\ell)}$

### Valued Quiver Representations:

A representation  $V = (\{V_i\}_{i \in Q_0}, \{\varphi_a\}_{a \in Q_1})$  of  $(Q, \mathbf{d})$  consists of an  $\mathbb{F}_{d_i}$ -vector space  $V_i$  for each vertex i and an  $\mathbb{F}_{gcd}(d_{s(a)}, d_{t(a)})$ -linear map  $\varphi_a : V_{s(a)} \to V_{t(a)}$  for each arrow a.

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 rep<sub>ℙ</sub>(Q, d) - hereditary, Abelian category of finite dimensional representations of (Q, d) (equivalent to modules over a species)

April 20, 2013

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- $\alpha_i$  isomorphism class of the vertex-simple  $S_i$
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Suppose  $V, W \in \operatorname{rep}_{\mathbb{F}}(Q, \mathbf{d})$ . We will need the Euler-Ringel form given by  $\langle V, W \rangle = \dim_{\mathbb{F}} \operatorname{Hom}(V, W) - \dim_{\mathbb{F}} \operatorname{Ext}^{1}(V, W)$ .

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# Let $V \in \operatorname{rep}_{\mathbb{F}}(Q, \mathbf{d})$ and write $\mathbf{v} \in \mathcal{K}(Q, \mathbf{d})$ for the dimension vector of V.

### Quantum Cluster Character:

We define the quantum cluster character  $V \mapsto X_V \in \mathcal{T}_{\Lambda,q}$  by

$$X_{V} = \sum_{\mathbf{e} \in \mathcal{K}(Q, \mathbf{d})} q^{-\frac{1}{2} \langle \mathbf{e}, \mathbf{v} - \mathbf{e} \rangle} |Gr_{\mathbf{e}}(V)| X^{-\mathbf{e}^{*} - *(\mathbf{v} - \mathbf{e})}$$

where  $Gr_{e}(V)$  denotes the Grassmannian of subrepresentations of V with isomorphism class e.

# Theorem (R.)

The quantum cluster character  $V \mapsto X_V$  defines a bijection from indecomposable rigid representations V of  $(Q, \mathbf{d})$  to non-initial quantum cluster variables of the quantum cluster algebra  $\mathcal{A}_q(\tilde{B}, \Lambda) \subset \mathcal{T}_{\Lambda,q}$ .

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### Quantum Cluster Character:

We define the quantum cluster character  $V \mapsto X_V \in \mathcal{T}_{\Lambda,q}$  by

$$X_{V} = \sum_{\mathbf{e} \in \mathcal{K}(Q, \mathbf{d})} q^{-\frac{1}{2} \langle \mathbf{e}, \mathbf{v} - \mathbf{e} \rangle} |Gr_{\mathbf{e}}(V)| X^{-\mathbf{e}^{*} - *(\mathbf{v} - \mathbf{e})}$$

where  $Gr_{\mathbf{e}}(V)$  denotes the Grassmannian of subrepresentations of V with isomorphism class  $\mathbf{e}$ .

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#### External Mutations

Reflection Functors and Quantum Cluster Characters

- rep<sub>𝔅</sub>(Q, d)⟨k⟩ full subcategory of rep<sub>𝔅</sub>(Q, d) of objects without indecomposable summands isomorphic to the simple S<sub>k</sub>
- $\mu_k Q$  quiver obtained from Q by reversing all arrows incident on vertex k
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# Two Perspectives on Mutations External Mutations Reflection Functors and Quantum Cluster Characters

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For any vertex k, the cluster variable obtained from the initial cluster by mutating in direction k is given by the quantum cluster character  $X_{S_k}$ .

A cluster  $(\mathbf{X}', \tilde{B}', \Lambda')$  is called almost acyclic if there exists a vertex k so that  $(\mu_k \mathbf{X}', \mu_k \tilde{B}', \mu_k \Lambda')$  is acyclic.

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Any cluster variable of  $\mathcal{A}_q(\tilde{B}, \Lambda)$  in an almost acyclic cluster is given by  $X_V$  for some representation V which can be obtained via reflection functors from a simple representation.

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- V is rigid if  $Ext^1(V, V) = 0$ .
- *V* is basic if each indecomposable summand appears with multiplicity one.
- The support of V is the set supp $(V) = \{i \in Q_0 : V_i \neq 0\}.$
- V is sincere if  $supp(V) = Q_0$ .

### Local Tilting Representations:

We will call a representation  $T \in \operatorname{rep}_{\mathbb{F}}(Q, \mathbf{d})$  local tilting if T is basic, rigid, and the number of indecomposable summands is equal to the number of vertices in its support.

Important: the zero representation is local tilting.

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To make this precise we will need to recall two classical theorems on tilting in hereditary categories:

### Theorem (Happel, Ringel)

Suppose T is basic and rigid. Then T is a tilting representation if and only if T has as many non-isomorphic indecomposable summands as the number of simple representations.

This allows us to restrict a local tilting representation T to the full subquiver of  $(Q, \mathbf{d})$  on the vertices supp(T) where it becomes a tilting representation.

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└─ Mutations of Local Tilting Representations

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Let T be an almost complete tilting representation. If T is sincere, then there exist exactly two non-isomorphic complements to T, otherwise there is a unique complement.

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## Mutation of Local Tilting Representations:

### Define the mutation $\mu_k(T) = T'$ in direction k as follows:

If vertex k ∉ supp(T), then there exists a unique complement T'<sub>k</sub> so that T' = T'<sub>k</sub> ⊕ T is a local tilting representation containing k in its support.

**2** If vertex  $k \in \text{supp}(T)$ , then write  $\overline{T} = T/T_k$ .

If T is a local tilting representation, i.e. k ∉ supp(T), let T' = T.
 Otherwise supp(T) = supp(T) and there exists a unique compliment T'<sub>k</sub> ∉ T<sub>k</sub> so that T' = T'<sub>k</sub> ⊕ T is a local tilting representation.

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Relationship to Quantum Cluster Algebras

We assign a quantum seed  $\Sigma_{T} = (\mathbf{X}_{T}, \tilde{B}_{T}, \Lambda_{T})$  to each local tilting representation T as follows:

•  $\mathbf{X}_{\mathcal{T}} = (X'_1, \dots, X'_m)$  is given by

$$X'_{k} = \begin{cases} X_{k} & \text{if } k \notin \text{supp}(T); \\ X_{T_{k}} & \text{if } k \in \text{supp}(T); \end{cases}$$

 The k<sup>th</sup> column of the exchange matrix B
<sub>T</sub> is defined homologically in terms of T and T<sup>\*</sup><sub>k</sub> (Hubery);

•  $\Lambda_{T}$  records the quasi-commutation of  $X_{T}$  (explicitly given by formulas involving the Euler-Ringel form and  $\Lambda$ ).

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  - Relationship to Quantum Cluster Algebras

#### Theorem (R.)

Suppose  $\mu_k(T) = T'$ . Then  $\Sigma_T$  and  $\Sigma_{T'}$  are related by Berenstein-Zelevinsky quantum seed mutation in direction k.

#### Lemma

The quantum seed associated to the zero representation is exactly the initial quantum seed  $(\mathbf{X}, \tilde{B}, \Lambda)$ .

#### Corollary (R.)

The quantum cluster character  $V \mapsto X_V$  defines a bijection from indecomposable rigid representations of  $(Q, \mathbf{d})$  to non-initial quantum cluster variables of the quantum cluster algebra  $\mathcal{A}_q(\tilde{B}, \Lambda)$ .

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# Thank you!

Dylan Rupel (NEU)

**Two Perspectives on Mutations** 

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