The Auslander bijections.

- Auslander, M.: *Functors and morphisms determined by objects.* In: Representation Theory of Algebras. Lecture Notes in Pure Appl. Math. 37. Marcel Dekker **(1978), 1-244.**
  Also in: Selected Works of Maurice Auslander, AMS (1999).

- Auslander, M.: *Applications of morphisms determined by objects.* In: Representation Theory of Algebras. Lecture Notes in Pure Appl. Math. 37. Marcel Dekker **(1978), 245-327.**
  Also in: Selected Works of Maurice Auslander, AMS (1999).

- Auslander, M., Reiten, I., Smalø, S.: *Representation Theory of Artin Algebras.* Cambridge Studies in Advanced Mathematics 36. Cambridge University Press. 1997. Chapter XI.

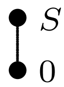
- Ringel, C. M.: *The Auslander bijections: How morphisms are determined by modules.* Bulletin of Mathematical Sciences (to appear). arXiv:1301.1251

Before we start: Submodule lattices.

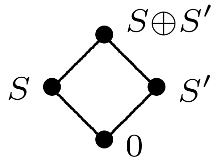
M a Λ-module of finite length, $\mathcal{S}M$ its submodule lattice.

Examples: $\mathcal{S}M$

S simple module



$S \oplus S'$, S, S' simple, $S \not\simeq S'$



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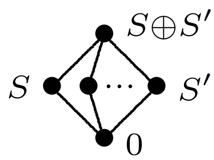
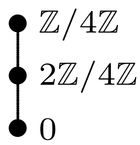


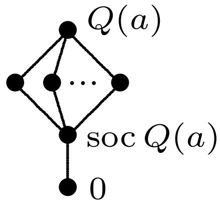
$\mathbb{Z}/4\mathbb{Z}$



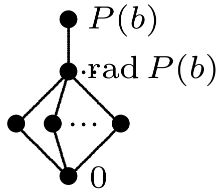
Λ Kronecker algebra 

$\mathcal{S}M$

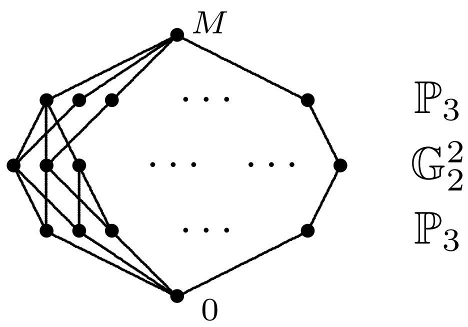
$M = Q(a)$ (injective)



$M = P(b)$ (projective)



$\Lambda = k, \quad M = k^4$



The Auslander bijections.

Λ artin algebra, $\operatorname{mod}\Lambda$ the left Λ-modules of finite length

$[\to Y\rangle$ the set of right equivalence classes of maps ending in Y

Definition: $f \preceq f' \iff f = f'h$ for some h.

Call $[f\rangle = \{f' \mid f \preceq f' \preceq f\}$ the right equivalence class of f

It is a poset, even a lattice.

$[f\rangle \leq [f'\rangle \iff f = f'h$

Any right equivalence class contains a right minimal map f

Recall that this means: $f\colon X \to Y$, $X = X' \oplus X''$, $f(X'') = 0 \implies X'' = 0$

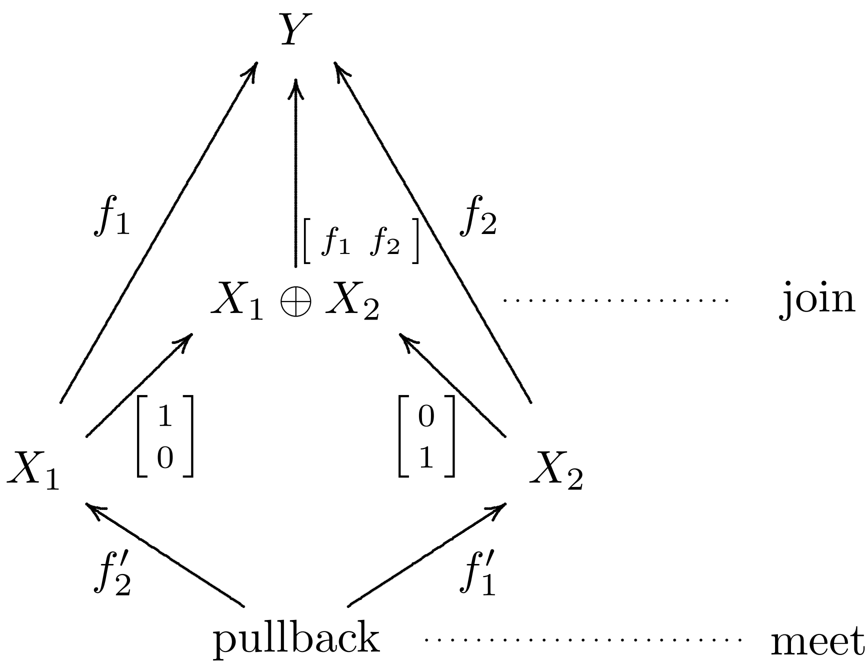
${}^C[\to Y\rangle$ the subset of all $[f\rangle$ with f "right C-determined".

In the special case when C is a generator:
f right C-determined $\iff$ $\operatorname{Ker} f \in \operatorname{add} \tau C$

($\tau = D\operatorname{Tr}$ the Auslander-Reiten translation.)

The lattice structure of $[\to Y\rangle$.

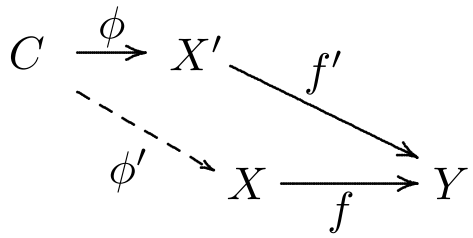
Given $f_1\colon X_1 \to Y,\ f_2\colon X_2 \to Y$.

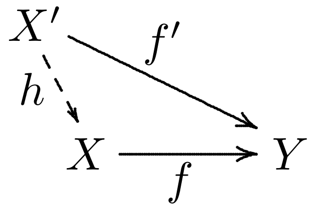


Remark: Even if f_1, f_2 are right minimal,
the maps $[f_1 f_2]$ and $f_2' f_1$ usually are not right minimal.

Right C-determination. $f\colon X \to Y$ morphism, C a module.

f is *right C-determined* provided any $f'\colon X' \to Y$ such that $f'\phi$ factors through f for all $\phi\colon C \to X'$, factors through f.





If $\operatorname{add} C = \operatorname{add} C'$, then f right C-determined iff f right C'-determined.

f right C-determined $\implies$ f right $(C \oplus C')$-determined.

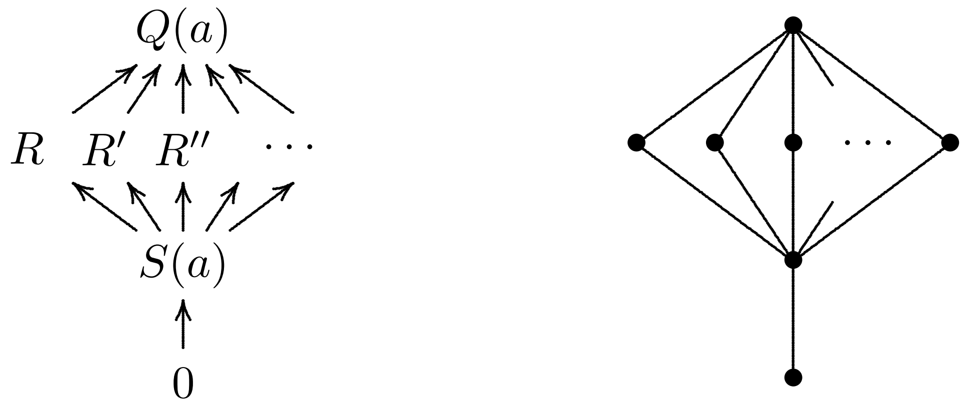
The subset ${}^{C}[\to Y\rangle$ *of* $[\to Y\rangle$ *is closed under meets.*

Warning. Usually it is not closed under joins.

It may not be advisable to look for subsets of $[\to Y\rangle$ closed under joins. The closure under joins may become very large!

For $C = \Lambda$, the lattice ${}^C[\to Y\rangle$ is just the submodule lattice $\mathcal{S}Y$ of Y.

Example: $\Lambda =$ Kronecker algebra with sink a, let $Y = Q(a)$.



$R, R', R'', \cdots$: the indecomposable representations of length 2 (note: the join in ${}^C[\to Y\rangle$ of maps $f_1 \neq f_2$ in the height 2 layer is $1\colon Y \to Y$),
the join in $[\to Y\rangle$ of pairwise different maps $f_i\colon R_i \to Y$ is the direct sum map $[f_1, \dots, f_n]\colon R_1 \oplus \cdots \oplus R_n \to Y$ (and these maps are right minimal).

If $|k| = \infty$, the smallest subposet of $[\to Y\rangle$ closed under joins and containing the inclusion maps $R \to Y$ (with R regular of length 2) has infinite height.

Auslander's First Theorem.

$$[\to Y\rangle = \bigcup_C {}^C[\to Y\rangle,$$

where C runs through all the Λ-modules
or through representatives of all multiplicity-free generators.

This is a filtered union of meet-semilattices.

If M is a module, let $\mathcal{S}M$ be the lattice of all submodules.
Consider $\mathrm{Hom}(C,Y)$ as a $\Gamma(C)$-module, where $\Gamma(C) = \mathrm{End}(C)^{\mathrm{op}}$.

Second Theorem. There is a poset isomorphism

$$\eta_{CY}\colon\ {}^C[\to Y\rangle \longrightarrow \mathcal{S}\,\mathrm{Hom}(C,Y).$$

with $\eta_{CY}(f) = \mathrm{Im}\,\mathrm{Hom}(C,f) = f\cdot\mathrm{Hom}(C,X)$
$= \{h \in \mathrm{Hom}(C,Y) \mid h \text{ factors through } f\}$.

Transfer from $\mathcal{S}\,\mathrm{Hom}(C,Y)$ to ${}^{C}[\to Y\rangle$.

$$\boxed{\eta_{CY}\colon\ {}^{C}[\to Y\rangle \longrightarrow \mathcal{S}\,\mathrm{Hom}(C,Y).}$$

$\mathcal{S}\,\mathrm{Hom}(C,Y)$ is a modular lattice of finite height, thus also ${}^{C}[\to Y\rangle$ is a **modular lattice of finite height.**

$\mathcal{S}\,\mathrm{Hom}(C,Y)$ has two distinguished elements: zero and one.

one	1_Y	$\mathrm{Hom}(C,Y)$
zero	$\eta_{CY}^{-1}(0) = ??$	0

Jordan-Hölder Theorem (composition series and factors) — my lecture today.

Krull-Remak-Schmidt Theorem (indecomposable summands) — the corresponding theory for ${}^{C}[\to Y\rangle$ is not yet clear.

Quiver Grassmannian ($\mathcal{S}M$ considered as algebraic variety) — next week!

The Jordan-Hölder Theorem for ${}^{C}[\to Y\rangle$.

Height of the lattice ${}^{C}[\to Y\rangle$
= length of a maximal chain of non-invertible maps

$$X_n \to X_{n-1} \to \cdots \to X_1 \to X_0 = Y$$

with all $X_i \to \cdots \to X_0 = Y$ right minimal, right C-determined.

Let $h_i \colon X_i \to X_{i-1}$ be maps with composition $f = h_1 \dots h_t$. $(h_1, h_2, \dots, h_t)$ is called a *right C-factorization of f of length t* iff all h_i are non-invertible and
all $f_i = h_i \cdots h_1$ are right minimal, right C-determined.

A right C-factorization $(h_1, h_2, \dots, h_t)$ is *maximal* provided it has no proper refinement.

Proposition *Any right C-factorization $(h_1, \dots, h_t)$ has a refinement which is a maximal right C-factorization and all maximal right C-factorizations of $(h_1, \dots, h_t)$ have the same length.*

In particular: any right minimal right C-determined map f has a refinement which is a maximal right C-factorization, its length t will be called the *C-length* of f, written $|f|_C$.

Proposition *Let $f\colon X \to Y$ be right minimal and right C-determined. Then*

$$\boxed{|f|_C = |\operatorname{Hom}(C,Y)| - |\eta_{CY}(f)|,}$$

Here, $|\operatorname{Hom}(C,Y)|$ is the length of $\operatorname{Hom}(C,Y)$ as $\Gamma(C)$-module, and $|\eta_{CY}(f)|$ the length of its submodule $\eta_{CY}(f)$.

Proof: $|\operatorname{Hom}(C,Y)|$ is the height of $[1_Y\rangle$, and $|\eta_{CY}(f)|$ is the height of $[f\rangle$ in ${}^C[\to Y\rangle$.

The C-length of a map f. Two special cases for C.

Case 1. C projective.

The right minimal, right C-determined maps $f\colon X \to Y$ are (up to right equivalence) just the inclusion maps of submodules X of Y such that the socle of Y/X is generated by C.

If $f\colon X \to Y$ is right minimal and right C-determined, then f is injective and

$$\boxed{|f|_C = \sum_{\substack{S \text{with} \\ P(S)|C}} [\mathrm{Cok}(f) : S].}$$

$\eta_{CY}^{-1}(0)$ is the inclusion $X \to Y$, where X is the intersection of the kernels of all maps $Y \to Q(S)$, with S simple, $P(S)|C$.

The C-length of a map f. Two special cases for C.

Case 2. Assume that K has semisimple endomorphism ring.

Let $C = \tau^{-}K$ and assume that $\mathcal{P}(C, Y) = 0$.
$f\colon X \to Y$ right min., right C-det. $\implies$ f is surjective and

$$\boxed{|f|_C = \mu(\mathrm{Ker}(f)).}$$

$\eta_{CY}^{-1}(0)$ is given by universal extension
$0 \to K' \to X \to Y \to 0$ with $K' \in \operatorname{add} K$.

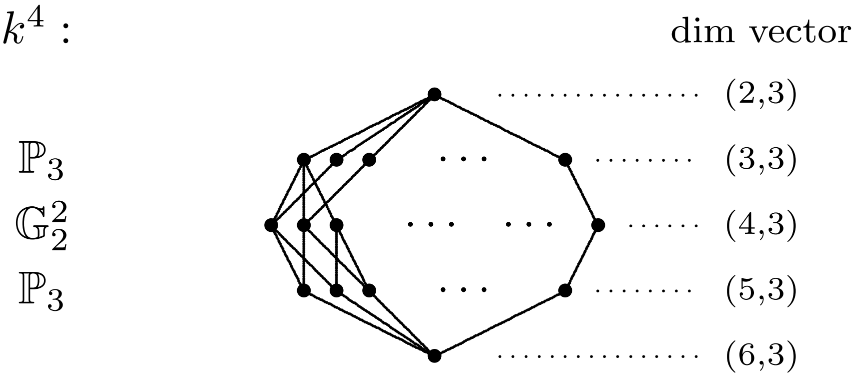
Example. Λ the Kronecker algebra,
$Y = (2, 3)$ (preinjective);
$C = (3, 2)$ (preprojective), thus $K = (1, 0)$.

$\mathrm{End}(C) = k$, thus $\mathcal{S}\,\mathrm{Hom}(C, Y)$ is the subspace set of $\mathrm{Hom}(C, Y)$.

$\mathrm{Hom}(C, Y) = k^4$, thus we deal with the geometry of $\mathbb{P}_3$.

Λ Kronecker algebra, $\mathbf{dim}\, Y = (2,3)$, $\mathbf{dim}\, C = (3,2)$.

${}^{C}[\to Y\rangle \simeq \mathcal{S}k^4$:



Height 4: 1_Y
Height 3: multiplicity-free regular modules of length 6,
Height 2: modules with dimension vector $(4,3)$,
both indecomposables, as well as decomposables.
Height 1: modules with dimension vector $(5,3)$,
both indecomposables, as well as decomposables.
Height 0: the projective cover of Y.

The C-type of C-neighbors.

f, f' right minimal, right C-determined maps with $[f\rangle \le [f'\rangle$.

Call (f, f') *C-neighbors* provided $|f|_C = |f'|_C + 1$,
provided $[f\rangle < [f'\rangle$ and there is no f'' with $[f\rangle < [f''\rangle < [f'\rangle$

Let (f, f') be C-neighbors, $f\colon X \to Y$, $f'\colon X' \to Y$.
(f, f') is *of type* C_0, provided C_0 an indecomposable direct summand of C and there is $\phi\colon C_0 \to X'$ such that $f'\phi$ does not factor through f.

(Such a summand C_0 must exist, since otherwise f' would factor through f, since f is right C_0-determined.)

Proposition *If (f, f') is of type C_0, then as $\Gamma(C)$-modules:*

$$\boxed{\eta_{CY}(f')/\eta_{CY}(f) \simeq \operatorname{top}\operatorname{Hom}(C, C_0)}$$

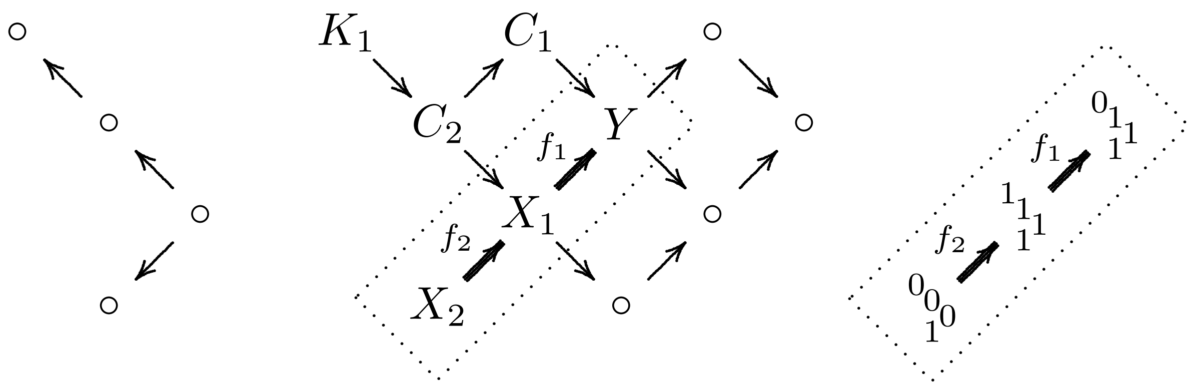
Note that $\operatorname{Hom}(C, C_0)$ is a simple $\Gamma(C)$-module.

Proposition *If* (f, f') *is of type* C_0*, then as* $\Gamma(C)$*-modules:*

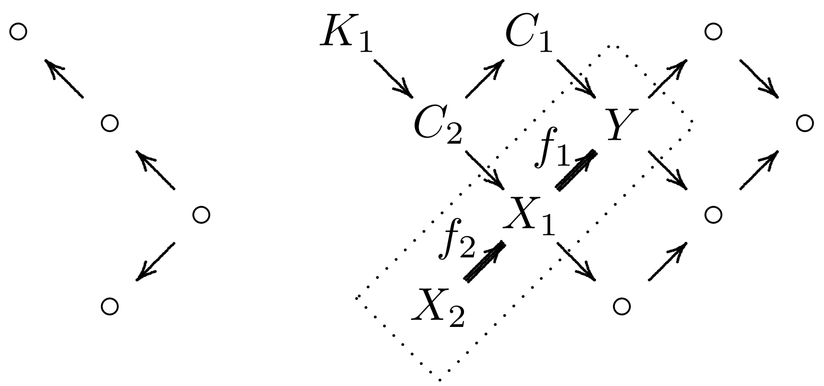
$$\boxed{\eta_{CY}(f')/\eta_{CY}(f) \simeq \operatorname{top}\operatorname{Hom}(C, C_0)}$$

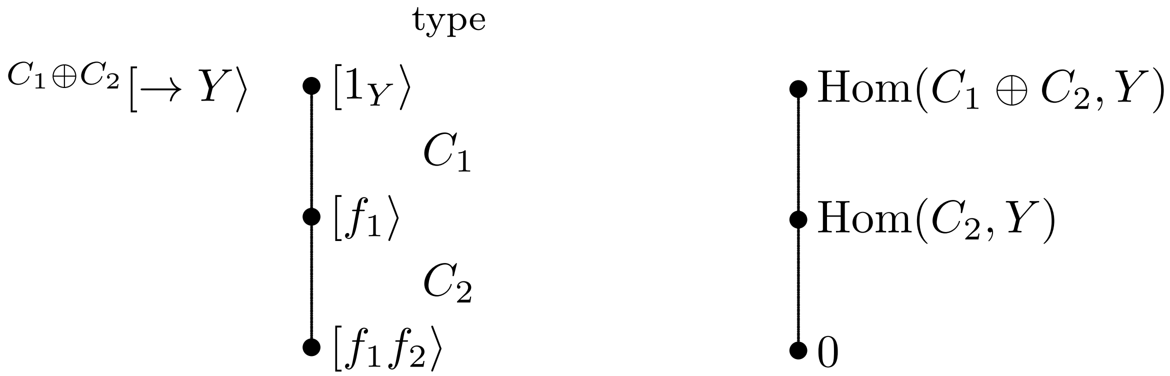
Corollary *The type of a pair of* C*-neighbors is uniquely determined.*

Example. We consider the quiver of type $\mathbb{A}_4$ with two sinks and one source.



f_1 is surjective with kernel K_1,
$f_1 f_2$ is injective, the socle of the cokernel is generated by C_2


K_1
C_1
C_2
Y
f_1
X_1
f_2
X_2

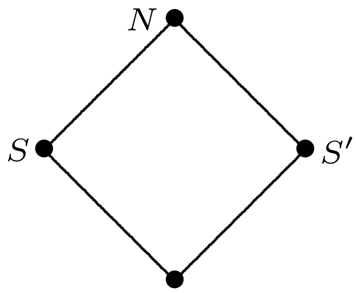

type
${}^{C_1 \oplus C_2}[\to Y\rangle$
$[1_Y\rangle$
C_1
$[f_1\rangle$
C_2
$[f_1 f_2\rangle$
$\mathrm{Hom}(C_1 \oplus C_2, Y)$
$\mathrm{Hom}(C_2, Y)$
0

Intervals of height 2.

Consider intervals of height 2 which are not linearly ordered.

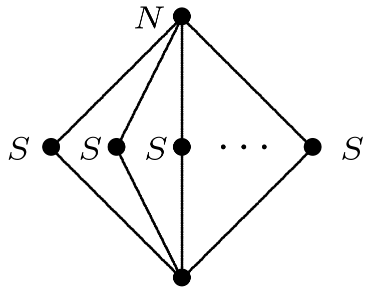
Such an interval in a submodule lattice $\mathcal{S}M$ is of the form $\mathcal{S}N$ with $N = S \oplus S'$ and simple modules S, S'.
There are two possibilities:

I. No diagonals



$S \not\simeq S'$

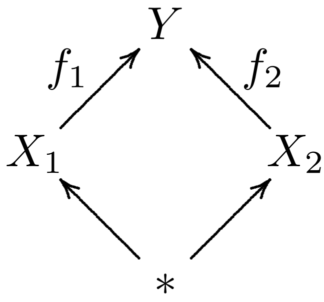
II. With diagonals



$S = S'$

The number of elements of height 1 is $1+|\operatorname{End}(S)|$

Corresponding pictures for $^{C}[\to Y\rangle$:

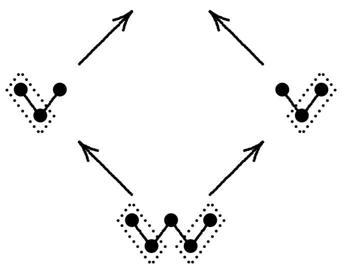


I. No diagonals.

The modules X_1, X_2 may be isomorphic or non-isomorphic!

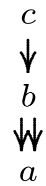
Example 1.
Λ the Kronecker quiver

$^{C}[\to Y\rangle$ $S(b)$

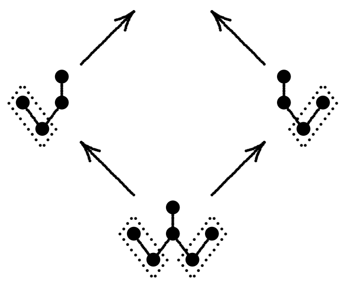


$C = R(0) \oplus R(\infty)$

Example 2.
Λ extended Kron. quiver



$^{C}[\to Y\rangle$ $Q(b)$

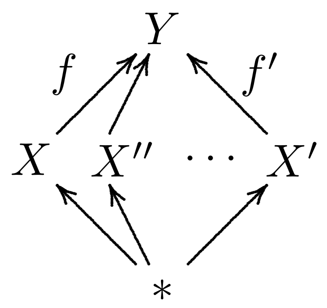


$C = R(0) \oplus R(\infty)$

What matters are not X_1, X_2, but the C-types of f_1, f_2.

Here, the C-types are non-isomorphic.

Corresponding pictures for ${}^C[\to Y\rangle$:

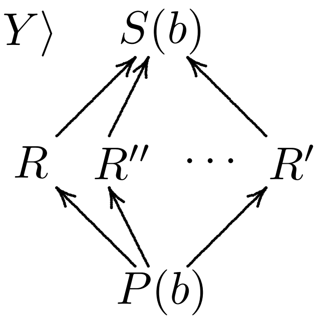


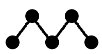
II. With diagonals.

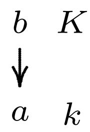
Again, the modules $X, X', X'', \ldots$
may be isomorphic or non-isomorphic!

Example 1.
Λ the Kronecker quiver 

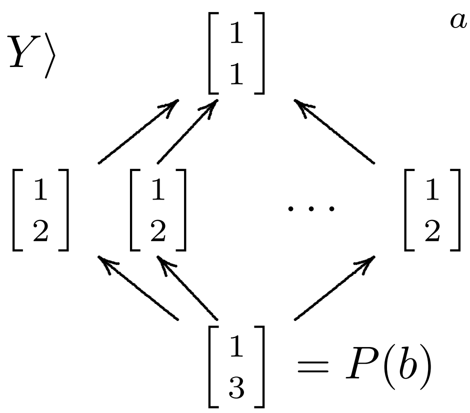
${}^C[\to Y\rangle$



$C =$ 

Example 2.
Λ hereditary of type $\mathbb{G}_2$ 

${}^C[\to Y\rangle$



$C = P(a) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Again, what matters are not X, X', but the C-types of f, f'.

Here, the C-types are isomorphic!