Bruhat order, Ext-quiver and Verma-multiplicities

Daiva Pučinskaitė

University of Kiel

Maurice Auslander Distinguished Lectures and International Conference

Woods Hole

April 18, 2013

Overview

Overview

$\boxed{\text{Any algebra is a finite dimensional } \mathbb{C}\text{-algebra}}$

Overview

$\boxed{\text{Any algebra is a finite dimensional } \mathbb{C}\text{-algebra}}$

$\mathfrak{g}$ is a semi simple
Lie algebra

Overview

$\boxed{\text{Any algebra is a finite dimensional } \mathbb{C}\text{-algebra}}$

$\mathfrak{g}$ is a semi simple
Lie algebra

$\mathcal{O}_\lambda(\mathfrak{g})$

Overview

$\boxed{\text{Any algebra is a finite dimensional } \mathbb{C}\text{-algebra}}$

$\mathfrak{g}$ is a semi simple
Lie algebra

$$\mathcal{O}_\lambda(\mathfrak{g}) \sim \operatorname{mod} A_\lambda(\mathfrak{g})$$

Overview

$\boxed{\text{Any algebra is a finite dimensional } \mathbb{C}\text{-algebra}}$

$\mathfrak{g}$ is a semi simple Lie algebra

$A_\lambda(\mathfrak{g}) = \mathbb{C}Q/\mathcal{I}$ is a bound quiver algebra

$$\mathcal{O}_\lambda(\mathfrak{g}) \sim \operatorname{mod} A_\lambda(\mathfrak{g})$$

Overview

$\boxed{\text{Any algebra is a finite dimensional } \mathbb{C}\text{-algebra}}$

$\mathfrak{g}$ is a semi simple Lie algebra

$A_\lambda(\mathfrak{g}) = \mathbb{C}Q/\mathcal{I}$ is a bound quiver algebra

$\mathcal{O}_\lambda(\mathfrak{g}) \sim \text{mod}\, A_\lambda(\mathfrak{g})$

$Q = (Q_{\text{ver}}, Q_{\text{arr}})$

Overview

$\boxed{\text{Any algebra is a finite dimensional } \mathbb{C}\text{-algebra}}$

$\mathfrak{g}$ is a semi simple Lie algebra

$A_\lambda(\mathfrak{g}) = \mathbb{C}Q/\mathcal{I}$ is a bound quiver algebra

$\mathcal{O}_\lambda(\mathfrak{g}) \sim \operatorname{mod} A_\lambda(\mathfrak{g})$

$Q = (Q_{\text{ver}}, Q_{\text{arr}})$

Weyl group $W \rightsquigarrow Q_{\text{ver}}$

Overview

$\boxed{\text{Any algebra is a finite dimensional } \mathbb{C}\text{-algebra}}$

$\mathfrak{g}$ is a semi simple Lie algebra

$A_\lambda(\mathfrak{g}) = \mathbb{C}Q/\mathcal{I}$ is a bound quiver algebra

$$\mathcal{O}_\lambda(\mathfrak{g}) \quad \sim \quad \operatorname{mod} A_\lambda(\mathfrak{g}) \qquad Q = (Q_{\mathsf{ver}}, Q_{\mathsf{arr}})$$

Weyl group $W \quad \leftrightsquigarrow \quad Q_{\mathsf{ver}}$

Bruhat order $\leqslant \quad \leftrightsquigarrow \quad Q_{\mathsf{arr}}$

Overview

$\boxed{\text{Any algebra is a finite dimensional } \mathbb{C}\text{-algebra}}$

$\mathfrak{g}$ is a semi simple Lie algebra

$A_\lambda(\mathfrak{g}) = \mathbb{C}Q/\mathcal{I}$ is a bound quiver algebra

$$\mathcal{O}_\lambda(\mathfrak{g}) \sim \operatorname{mod} A_\lambda(\mathfrak{g}) \qquad Q = (Q_{\mathsf{ver}}, Q_{\mathsf{arr}})$$

$$\begin{array}{lcl} \text{Weyl group } W & \leftrightsquigarrow & Q_{\mathsf{ver}} \\ \text{Bruhat order } \leqslant & \leftrightsquigarrow & Q_{\mathsf{arr}} \end{array}$$

$\boxed{A_\lambda(\mathfrak{g}) \text{ is quasi-hereditary}}$

Overview

$\boxed{\text{Any algebra is a finite dimensional } \mathbb{C}\text{-algebra}}$

$\mathfrak{g}$ is a semi simple Lie algebra

$A_\lambda(\mathfrak{g}) = \mathbb{C}Q/\mathcal{I}$ is a bound quiver algebra

$$\mathcal{O}_\lambda(\mathfrak{g}) \sim \operatorname{mod} A_\lambda(\mathfrak{g}) \qquad Q = (Q_{\text{ver}}, Q_{\text{arr}})$$

$$\begin{array}{lcl} \text{Weyl group } W & \leftrightsquigarrow & Q_{\text{ver}} \\ \text{Bruhat order } \leqslant & \leftrightsquigarrow & Q_{\text{arr}} \end{array}$$

$\boxed{A_\lambda(\mathfrak{g}) \text{ is quasi-hereditary}}$

$0 \subset \cdots \subset M_i \subset M_{i+1} \subset \cdots \subset P(w)$ Verma filtration

Overview

$\boxed{\text{Any algebra is a finite dimensional } \mathbb{C}\text{-algebra}}$

$\mathfrak{g}$ is a semi simple Lie algebra

$A_\lambda(\mathfrak{g}) = \mathbb{C}Q/\mathcal{I}$ is a bound quiver algebra

$$\mathcal{O}_\lambda(\mathfrak{g}) \sim \operatorname{mod} A_\lambda(\mathfrak{g}) \qquad Q = (Q_{\text{ver}}, Q_{\text{arr}})$$

Weyl group $W \rightsquigarrow Q_{\text{ver}}$

Bruhat order $\leqslant \rightsquigarrow Q_{\text{arr}}$

$\boxed{A_\lambda(\mathfrak{g}) \text{ is quasi-hereditary}}$

$$0 \subset \cdots \subset \underbrace{M_i \subset M_{i+1}}_{\Delta(v)} \subset \cdots \subset P(w) \quad \text{Verma filtration}$$

Overview

$\boxed{\text{Any algebra is a finite dimensional } \mathbb{C}\text{-algebra}}$

$\mathfrak{g}$ is a semi simple Lie algebra

$A_\lambda(\mathfrak{g}) = \mathbb{C}Q/\mathcal{I}$ is a bound quiver algebra

$\mathcal{O}_\lambda(\mathfrak{g}) \sim \operatorname{mod} A_\lambda(\mathfrak{g})$

$Q = (Q_{\text{ver}}, Q_{\text{arr}})$

Weyl group $W \rightsquigarrow Q_{\text{ver}}$

Bruhat order $\leqslant \rightsquigarrow Q_{\text{arr}}$

$\boxed{A_\lambda(\mathfrak{g}) \text{ is quasi-hereditary}}$

$0 \subset \cdots \subset \underbrace{M_i \subset M_{i+1}}_{\Delta(v)} \subset \cdots \subset P(w)$ Verma filtration

BGG Category $\mathcal{O}(\mathfrak{g})$

BGG Category $\mathcal{O}(\mathfrak{g})$

- $\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-$ is a Cartan decomposition

BGG Category $\mathcal{O}(\mathfrak{g})$

- $\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-$ is a Cartan decomposition
- $\mathfrak{U}(\mathfrak{g})$ is the universal enveloping algebra of $\mathfrak{g}$

BGG Category $\mathcal{O}(\mathfrak{g})$

- $\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-$ is a Cartan decomposition
- $\mathfrak{U}(\mathfrak{n}_+)$ is the universal enveloping algebra of $\mathfrak{n}_+$

BGG Category $\mathcal{O}(\mathfrak{g})$

- $\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-$ is a Cartan decomposition
- $\mathfrak{U}(\mathfrak{n}_+)$ is the universal enveloping algebra of $\mathfrak{n}_+$

Definition (Bernstein, Gelfand, Gelfand)

BGG Category $\mathcal{O}(\mathfrak{g})$

- $\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-$ is a Cartan decomposition
- $\mathfrak{U}(\mathfrak{n}_+)$ is the universal enveloping algebra of $\mathfrak{n}_+$

Definition (Bernstein, Gelfand, Gelfand)

The category $\mathcal{O}(\mathfrak{g})$ is the full subcategory of $\mathfrak{g}$-modules M with

BGG Category $\mathcal{O}(\mathfrak{g})$

- $\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-$ is a Cartan decomposition
- $\mathfrak{U}(\mathfrak{n}_+)$ is the universal enveloping algebra of $\mathfrak{n}_+$

Definition (Bernstein, Gelfand, Gelfand)

The category $\mathcal{O}(\mathfrak{g})$ is the full subcategory of $\mathfrak{g}$-modules M with

- M is a finitely generated $\mathfrak{U}(\mathfrak{g})$-module,

BGG Category $\mathcal{O}(\mathfrak{g})$

- $\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-$ is a Cartan decomposition
- $\mathfrak{U}(\mathfrak{n}_+)$ is the universal enveloping algebra of $\mathfrak{n}_+$

Definition (Bernstein, Gelfand, Gelfand)

The category $\mathcal{O}(\mathfrak{g})$ is the full subcategory of $\mathfrak{g}$-modules M with

- M is a finitely generated $\mathfrak{U}(\mathfrak{g})$-module,
- M is $\mathfrak{h}$-semisimple,

BGG Category $\mathcal{O}(\mathfrak{g})$

- $\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-$ is a Cartan decomposition
- $\mathfrak{U}(\mathfrak{n}_+)$ is the universal enveloping algebra of $\mathfrak{n}_+$

Definition (Bernstein, Gelfand, Gelfand)

The category $\mathcal{O}(\mathfrak{g})$ is the full subcategory of $\mathfrak{g}$-modules M with

- M is a finitely generated $\mathfrak{U}(\mathfrak{g})$-module,
- M is $\mathfrak{h}$-semisimple,
- $\dim_{\mathbb{C}} [\mathfrak{U}(\mathfrak{n}_+) \cdot m] < \infty$ for all $m \in M$.

BGG Category $\mathcal{O}(\mathfrak{g})$

- $\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-$ is a Cartan decomposition
- $\mathfrak{U}(\mathfrak{n}_+)$ is the universal enveloping algebra of $\mathfrak{n}_+$

Definition (Bernstein, Gelfand, Gelfand)

The category $\mathcal{O}(\mathfrak{g})$ is the full subcategory of $\mathfrak{g}$-modules M with

- M is a finitely generated $\mathfrak{U}(\mathfrak{g})$-module,
- M is $\mathfrak{h}$-semisimple,
- $\dim_{\mathbb{C}} [\mathfrak{U}(\mathfrak{n}_+) \cdot m] < \infty$ for all $m \in M$.

- Important modules in $\mathcal{O}(\mathfrak{g})$

BGG Category $\mathcal{O}(\mathfrak{g})$

- $\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-$ is a Cartan decomposition
- $\mathfrak{U}(\mathfrak{n}_+)$ is the universal enveloping algebra of $\mathfrak{n}_+$

Definition (Bernstein, Gelfand, Gelfand)

The category $\mathcal{O}(\mathfrak{g})$ is the full subcategory of $\mathfrak{g}$-modules M with

- M is a finitely generated $\mathfrak{U}(\mathfrak{g})$-module,
- M is $\mathfrak{h}$-semisimple,
- $\dim_{\mathbb{C}} [\mathfrak{U}(\mathfrak{n}_+) \cdot m] < \infty$ for all $m \in M$.

- Important modules in $\mathcal{O}(\mathfrak{g})$

$\lambda \in \mathfrak{h}^*$

BGG Category $\mathcal{O}(\mathfrak{g})$

- $\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-$ is a Cartan decomposition
- $\mathfrak{U}(\mathfrak{n}_+)$ is the universal enveloping algebra of $\mathfrak{n}_+$

Definition (Bernstein, Gelfand, Gelfand)

The category $\mathcal{O}(\mathfrak{g})$ is the full subcategory of $\mathfrak{g}$-modules M with

- M is a finitely generated $\mathfrak{U}(\mathfrak{g})$-module,
- M is $\mathfrak{h}$-semisimple,
- $\dim_{\mathbb{C}} [\mathfrak{U}(\mathfrak{n}_+) \cdot m] < \infty$ for all $m \in M$.

- Important modules in $\mathcal{O}(\mathfrak{g})$

$$\begin{tikzcd} & \{ \text{ Verma module } \Delta(\lambda) \} \\ \lambda \in \mathfrak{h}^* \arrow[ur, leftrightarrow] & \end{tikzcd}$$

BGG Category $\mathcal{O}(\mathfrak{g})$

- $\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-$ is a Cartan decomposition
- $\mathfrak{U}(\mathfrak{n}_+)$ is the universal enveloping algebra of $\mathfrak{n}_+$

Definition (Bernstein, Gelfand, Gelfand)

The category $\mathcal{O}(\mathfrak{g})$ is the full subcategory of $\mathfrak{g}$-modules M with

- M is a finitely generated $\mathfrak{U}(\mathfrak{g})$-module,
- M is $\mathfrak{h}$-semisimple,
- $\dim_{\mathbb{C}} [\mathfrak{U}(\mathfrak{n}_+) \cdot m] < \infty$ for all $m \in M$.

- Important modules in $\mathcal{O}(\mathfrak{g})$

$$\begin{tikzcd} & \{ \text{ Verma module } \Delta(\lambda) \} \arrow[dl, leftrightarrow] \\ \lambda \in \mathfrak{h}^* \arrow[r, leftrightarrow] & \{ \text{ simple module } S(\lambda) \} \end{tikzcd}$$

BGG Category $\mathcal{O}(\mathfrak{g})$

- $\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-$ is a Cartan decomposition
- $\mathfrak{U}(\mathfrak{n}_+)$ is the universal enveloping algebra of $\mathfrak{n}_+$

Definition (Bernstein, Gelfand, Gelfand)

The category $\mathcal{O}(\mathfrak{g})$ is the full subcategory of $\mathfrak{g}$-modules M with

- M is a finitely generated $\mathfrak{U}(\mathfrak{g})$-module,
- M is $\mathfrak{h}$-semisimple,
- $\dim_{\mathbb{C}} [\mathfrak{U}(\mathfrak{n}_+) \cdot m] < \infty$ for all $m \in M$.

- Important modules in $\mathcal{O}(\mathfrak{g})$

$$\lambda \in \mathfrak{h}^* \longleftrightarrow \begin{cases} \{ \text{ Verma module } \Delta(\lambda) \ \} \\ \{ \text{ simple module } S(\lambda) \ \} \\ \{ \text{projective indec. } P(\lambda) \ \} \end{cases}$$

BGG Category $\mathcal{O}(\mathfrak{g})$

- $\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-$ is a Cartan decomposition
- $\mathfrak{U}(\mathfrak{n}_+)$ is the universal enveloping algebra of $\mathfrak{n}_+$

Definition (Bernstein, Gelfand, Gelfand)

The category $\mathcal{O}(\mathfrak{g})$ is the full subcategory of $\mathfrak{g}$-modules M with

- M is a finitely generated $\mathfrak{U}(\mathfrak{g})$-module,
- M is $\mathfrak{h}$-semisimple,
- $\dim_{\mathbb{C}} [\mathfrak{U}(\mathfrak{n}_+) \cdot m] < \infty$ for all $m \in M$.

- Important modules in $\mathcal{O}(\mathfrak{g})$

$$\begin{tikzcd}
 & \{\text{ Verma module } \Delta(\lambda)\ \} \arrow[l] \arrow[r, leftrightarrow] & \{\text{coVerma module } \nabla(\lambda)\} \\
\lambda \in \mathfrak{h}^* \arrow[ur] \arrow[r, leftrightarrow] \arrow[dr] & \{\text{ simple module } S(\lambda)\ \} & \\
 & \{\text{projective indec. } P(\lambda)\ \} \arrow[ul] \arrow[r, leftrightarrow] & \{\text{injective indec. } I(\lambda)\}
\end{tikzcd}$$

The blocks of $\mathcal{O}(\mathfrak{g})$

The blocks of $\mathcal{O}(\mathfrak{g})$

- $\mathfrak{h}^* = \bigcup_{\substack{\eta \in \mathfrak{h}^* \\ \eta \text{ is dominant}}} \mathfrak{h}^*_\eta$

The blocks of $\mathcal{O}(\mathfrak{g})$

- $\mathfrak{h}^* = \bigcup_{\substack{\eta \in \mathfrak{h}^* \\ \eta \text{ is dominant}}} \mathfrak{h}^*_\eta$ where $\mathfrak{h}^*_\eta$ is a finite set

The blocks of $\mathcal{O}(\mathfrak{g})$

- $\mathfrak{h}^* = \bigcup_{\substack{\eta \in \mathfrak{h}^* \\ \eta \text{ is dominant}}} \mathfrak{h}^*_\eta$ where $\mathfrak{h}^*_\eta (:= W_\eta \cdot \eta)$ is a finite set

The blocks of $\mathcal{O}(\mathfrak{g})$

- $\mathfrak{h}^* = \bigcup_{\substack{\eta\in\mathfrak{h}^* \\ \eta \text{ is dominant}}} \mathfrak{h}^*_\eta$ where $\mathfrak{h}^*_\eta (:= W_\eta \cdot \eta)$ is a finite set
- $\mathcal{O}_\eta(\mathfrak{g})$ is the full subcategory of modules M in $\mathcal{O}(\mathfrak{g})$ with

The blocks of $\mathcal{O}(\mathfrak{g})$

- $\mathfrak{h}^* = \bigcup_{\substack{\eta \in \mathfrak{h}^* \\ \eta \text{ is dominant}}} \mathfrak{h}^*_\eta$ where $\mathfrak{h}^*_\eta (:= W_\eta \cdot \eta)$ is a finite set
- $\mathcal{O}_\eta(\mathfrak{g})$ is the full subcategory of modules M in $\mathcal{O}(\mathfrak{g})$ with
$$[M : S(\lambda)] \neq 0$$

The blocks of $\mathcal{O}(\mathfrak{g})$

- $\mathfrak{h}^* = \bigcup_{\substack{\eta \in \mathfrak{h}^* \\ \eta \text{ is dominant}}} \mathfrak{h}^*_\eta$ where $\mathfrak{h}^*_\eta (:= W_\eta \cdot \eta)$ is a finite set
- $\mathcal{O}_\eta(\mathfrak{g})$ is the full subcategory of modules M in $\mathcal{O}(\mathfrak{g})$ with
$$[M : S(\lambda)] \neq 0 \implies \lambda \in \mathfrak{h}^*_\eta$$

The blocks of $\mathcal{O}(\mathfrak{g})$

- $$\mathfrak{h}^* = \bigcup_{\substack{\eta \in \mathfrak{h}^* \\ \eta \text{ is dominant}}} \mathfrak{h}^*_\eta \quad \text{where} \quad \mathfrak{h}^*_\eta (:= W_\eta \cdot \eta) \text{ is a finite set}$$
- $\mathcal{O}_\eta(\mathfrak{g})$ is the full subcategory of modules M in $\mathcal{O}(\mathfrak{g})$ with
$$[M : S(\lambda)] \neq 0 \implies \lambda \in \mathfrak{h}^*_\eta$$

$$\boxed{\mathcal{O}(\mathfrak{g}) = \bigoplus_{\substack{\eta \in \mathfrak{h}^* \\ \eta \text{ is dominant}}} \mathcal{O}_\eta(\mathfrak{g})}$$

The blocks of $\mathcal{O}(\mathfrak{g})$

- $\mathfrak{h}^* = \bigcup_{\substack{\eta \in \mathfrak{h}^* \\ \eta \text{ is dominant}}} \mathfrak{h}^*_\eta$ where $\mathfrak{h}^*_\eta (:= W_\eta \cdot \eta)$ is a finite set
- $\mathcal{O}_\eta(\mathfrak{g})$ is the full subcategory of modules M in $\mathcal{O}(\mathfrak{g})$ with
$$[M : S(\lambda)] \neq 0 \implies \lambda \in \mathfrak{h}^*_\eta$$

$$\boxed{\mathcal{O}(\mathfrak{g}) = \bigoplus_{\substack{\eta \in \mathfrak{h}^* \\ \eta \text{ is dominant}}} \mathcal{O}_\eta(\mathfrak{g})}$$

- Important modules in $\mathcal{O}(\mathfrak{g})$

$$\begin{array}{ccl} & \nearrow & \{\text{ Verma module } \Delta(\lambda)\ \} \longleftrightarrow \{\text{coVerma module } \nabla(\lambda)\} \\ \lambda \in \mathfrak{h}^* & \longleftrightarrow & \{\text{ simple module } S(\lambda)\ \} \\ & \searrow & \{\text{projective indec. } P(\lambda)\ \} \longleftrightarrow \{\text{injective indec. } I(\lambda)\} \end{array}$$

The blocks of $\mathcal{O}(\mathfrak{g})$

- $\mathfrak{h}^* = \bigcup_{\substack{\eta \in \mathfrak{h}^* \\ \eta \text{ is dominant}}} \mathfrak{h}^*_\eta$ where $\mathfrak{h}^*_\eta (:= W_\eta \cdot \eta)$ is a finite set
- $\mathcal{O}_\eta(\mathfrak{g})$ is the full subcategory of modules M in $\mathcal{O}(\mathfrak{g})$ with

$$[M : S(\lambda)] \neq 0 \implies \lambda \in \mathfrak{h}^*_\eta$$

$$\boxed{\mathcal{O}(\mathfrak{g}) = \bigoplus_{\substack{\eta \in \mathfrak{h}^* \\ \eta \text{ is dominant}}} \mathcal{O}_\eta(\mathfrak{g})}$$

- Important modules in $\mathcal{O}_\eta(\mathfrak{g})$

$$\begin{array}{ccl} & \nearrow & \{ \text{ Verma module } \Delta(\lambda) \ \} \longleftrightarrow \{\text{coVerma module } \nabla(\lambda)\} \\ \lambda \in \mathfrak{h}^*_\eta & \longleftrightarrow & \{ \text{ simple module } S(\lambda) \ \} \\ & \searrow & \{\text{projective indec. } P(\lambda) \ \} \longleftrightarrow \{\text{injective indec. } I(\lambda)\} \end{array}$$

The blocks of $\mathcal{O}(\mathfrak{g})$

- $\mathfrak{h}^* = \bigcup_{\substack{\eta \in \mathfrak{h}^* \\ \eta \text{ is dominant}}} \mathfrak{h}^*_\eta$ where $\mathfrak{h}^*_\eta (:= W_\eta \cdot \eta)$ is a finite set
- $\mathcal{O}_\eta(\mathfrak{g})$ is the full subcategory of modules M in $\mathcal{O}(\mathfrak{g})$ with
$$[M : S(\lambda)] \neq 0 \implies \lambda \in \mathfrak{h}^*_\eta$$

$$\boxed{\mathcal{O}(\mathfrak{g}) = \bigoplus_{\substack{\eta \in \mathfrak{h}^* \\ \eta \text{ is dominant}}} \mathcal{O}_\eta(\mathfrak{g})}$$

- Important modules in $\mathcal{O}_\eta(\mathfrak{g})$

$$\begin{array}{ccccc} & \nearrow & \{ \text{ Verma module } \Delta(\lambda) \ \} & \longleftrightarrow & \{\text{coVerma module } \nabla(\lambda)\} \\ \lambda \in \mathfrak{h}^*_\eta & \longleftrightarrow & \{ \text{ simple module } S(\lambda) \ \} & & \\ & \searrow & \{\text{projective indec. } P(\lambda) \ \} & \longleftrightarrow & \{\text{injective indec. } I(\lambda)\} \end{array}$$

- $$A_\eta(\mathfrak{g}) := \operatorname{End}_{\mathfrak{g}} \left(\bigoplus_{\lambda \in \mathfrak{h}^*_\eta} P(\lambda) \right)^{op}$$

The blocks of $\mathcal{O}(\mathfrak{g})$

- $\mathfrak{h}^* = \bigcup_{\substack{\eta\in\mathfrak{h}^* \\ \eta \text{ is dominant}}} \mathfrak{h}^*_\eta$ where $\mathfrak{h}^*_\eta (:= W_\eta \cdot \eta)$ is a finite set
- $\mathcal{O}_\eta(\mathfrak{g})$ is the full subcategory of modules M in $\mathcal{O}(\mathfrak{g})$ with
$$[M : S(\lambda)] \neq 0 \implies \lambda \in \mathfrak{h}^*_\eta$$

$$\boxed{\mathcal{O}(\mathfrak{g}) = \bigoplus_{\substack{\eta\in\mathfrak{h}^* \\ \eta \text{ is dominant}}} \mathcal{O}_\eta(\mathfrak{g})}$$

- Important modules in $\mathcal{O}_\eta(\mathfrak{g})$

$$\begin{array}{ccccc} & \nearrow & \{\text{ Verma module } \Delta(\lambda)\ \} & \longleftrightarrow & \{\text{coVerma module } \nabla(\lambda)\} \\ \lambda \in \mathfrak{h}^*_\eta & \longleftrightarrow & \{\text{ simple module } S(\lambda)\ \} & & \\ & \searrow & \{\text{projective indec. } P(\lambda)\ \} & \longleftrightarrow & \{\text{injective indec. } I(\lambda)\} \end{array}$$

- $\text{mod}\ [A_\eta(\mathfrak{g}) := \text{End}_{\mathfrak{g}} \left(\bigoplus_{\lambda\in\mathfrak{h}^*_\eta} P(\lambda)\right)^{op}] \sim \mathcal{O}_\eta(\mathfrak{g})$

The blocks of $\mathcal{O}(\mathfrak{g})$

- $\mathfrak{h}^* = \bigcup_{\substack{\eta \in \mathfrak{h}^* \\ \eta \text{ is dominant}}} \mathfrak{h}^*_\eta$ where $\mathfrak{h}^*_\eta (:= W_\eta \cdot \eta)$ is a finite set
- $\mathcal{O}_\eta(\mathfrak{g})$ is the full subcategory of modules M in $\mathcal{O}(\mathfrak{g})$ with

$$[M : S(\lambda)] \neq 0 \implies \lambda \in \mathfrak{h}^*_\eta$$

$$\boxed{\mathcal{O}(\mathfrak{g}) = \bigoplus_{\substack{\eta \in \mathfrak{h}^* \\ \eta \text{ is dominant}}} \mathcal{O}_\eta(\mathfrak{g})}$$

- Important modules in $\mathcal{O}_0(\mathfrak{g})$ (principal block)

$$\begin{array}{ccl} & \nearrow & \{ \text{ Verma module } \Delta(\lambda) \ \} \longleftrightarrow \{\text{coVerma module } \nabla(\lambda)\} \\ \lambda \in \mathfrak{h}^*_0 & \longleftrightarrow & \{ \text{ simple module } S(\lambda) \ \} \\ & \searrow & \{\text{projective indec. } P(\lambda) \ \} \longleftrightarrow \{\text{injective indec. } I(\lambda)\} \end{array}$$

- mod $[A_0(\mathfrak{g}) := \mathrm{End}_{\mathfrak{g}} \left(\bigoplus_{\lambda \in \mathfrak{h}^*_0} P(\lambda) \right)^{op}] \sim \mathcal{O}_0(\mathfrak{g})$

The blocks of $\mathcal{O}(\mathfrak{g})$

- $\mathfrak{h}^* = \bigcup_{\substack{\eta \in \mathfrak{h}^* \\ \eta \text{ is dominant}}} \mathfrak{h}^*_\eta$ where $\mathfrak{h}^*_\eta (:= W_\eta \cdot \eta)$ is a finite set
- $\mathcal{O}_\eta(\mathfrak{g})$ is the full subcategory of modules M in $\mathcal{O}(\mathfrak{g})$ with

$$[M : S(\lambda)] \neq 0 \implies \lambda \in \mathfrak{h}^*_\eta$$

$$\boxed{\mathcal{O}(\mathfrak{g}) = \bigoplus_{\substack{\eta \in \mathfrak{h}^* \\ \eta \text{ is dominant}}} \mathcal{O}_\eta(\mathfrak{g})}$$

- Important modules in $\mathcal{O}_0(\mathfrak{g})$ (principal block)

$$\begin{tikzcd}
 & \{ \text{ Verma module } \Delta(w) \ \} \arrow[r, leftrightarrow] & \{\text{coVerma module } \nabla(w)\} \\
w \in W \arrow[ur, leftrightarrow] \arrow[r, leftrightarrow] \arrow[dr, leftrightarrow] & \{ \text{ simple module } S(w) \ \} & \\
 & \{\text{projective indec. } P(w) \ \} \arrow[r, leftrightarrow] & \{\text{injective indec. } I(w)\}
\end{tikzcd}$$

- mod $[A_0(\mathfrak{g}) := \mathrm{End}_{\mathfrak{g}} \left(\bigoplus_{w \in W} P(w) \right)^{op}] \sim \mathcal{O}_0(\mathfrak{g})$

Overview

$\boxed{\text{Any algebra is a finite dimensional } \mathbb{C}\text{-algebra}}$

$\mathfrak{g}$ is a semi simple Lie algebra

$A_0(\mathfrak{g}) = \mathbb{C}Q/\mathcal{I}$ is a bound quiver algebra

$$\mathcal{O}_0(\mathfrak{g}) \sim \text{mod } A_0(\mathfrak{g}) \qquad Q = (Q_{\text{ver}}, Q_{\text{arr}})$$

$$\text{Weyl group } W \leftrightsquigarrow Q_{\text{ver}}$$

$$\text{Bruhat order } \leqslant \leftrightsquigarrow Q_{\text{arr}}$$

$\boxed{A_0(\mathfrak{g}) \text{ is quasi-hereditary}}$

$$0 \subset \cdots \subset \underbrace{M_i \subset M_{i+1}}_{\Delta(v)} \subset \cdots \subset P(w) \quad \text{Verma filtration}$$

Overview

$\boxed{\text{Any algebra is a finite dimensional } \mathbb{C}\text{-algebra}}$

$\mathfrak{g}$ is a semi simple Lie algebra

$A_0(\mathfrak{g}) = \mathbb{C}Q/\mathcal{I}$ is a bound quiver algebra

$\mathcal{O}_0(\mathfrak{g}) \sim \operatorname{mod} A_0(\mathfrak{g})$

$Q = (Q_{\text{ver}}, Q_{\text{arr}})$

$$\begin{array}{lcl} \text{Weyl group } W & \leftrightsquigarrow & Q_{\text{ver}} \\ \text{Bruhat order } \leqslant & \leftrightsquigarrow & Q_{\text{arr}} \end{array}$$

$\boxed{A_0(\mathfrak{g}) \text{ is quasi-hereditary}}$

$0 \subset \cdots \subset \underbrace{M_i \subset M_{i+1}}_{\Delta(v)} \subset \cdots \subset P(w)$ Verma filtration

Bruhat order on $W(\mathfrak{sl}_n) \cong \mathrm{Sym}(n)$

Bruhat order on $W(\mathfrak{sl}_n) \cong \mathrm{Sym}(n)$

$\mathrm{Sym}(n) = \{\text{permutations of } \{1, \ldots, n\}\}$

Bruhat order on $W(\mathfrak{sl}_n) \cong \mathrm{Sym}(n)$

$$\mathrm{Sym}(n) = \{\text{permutations of } \{1, \ldots, n\}\} = \langle (1,2), \ldots, (n-1, n) \rangle$$

Bruhat order on $W(\mathfrak{sl}_n) \cong \mathrm{Sym}(n)$

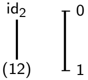
$\mathrm{Sym}(n) = \{\text{permutations of } \{1, \ldots, n\}\} = \langle (1,2), \ldots, (n-1, n) \rangle$

$l(\sigma) := \min \{m \; ; \; \sigma = \tau_1 \cdots \tau_m \text{ with } \tau_k \in \{(1,2), \ldots, (n-1, n)\}\}$

Bruhat order on $W(\mathfrak{sl}_n) \cong \mathrm{Sym}(n)$

$$\mathrm{Sym}(n) = \{\text{permutations of } \{1, \dots, n\}\} = \langle (1,2), \dots, (n-1, n) \rangle$$

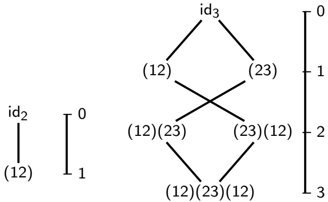
$$l(\sigma) := \min \{ m \; ; \; \sigma = \tau_1 \cdots \tau_m \text{ with } \tau_k \in \{(1,2), \dots, (n-1, n)\}\}$$



Bruhat order on $W(\mathfrak{sl}_n) \cong \mathrm{Sym}(n)$

$\mathrm{Sym}(n) = \{\text{permutations of } \{1, \dots, n\}\} = \langle (1,2), \dots, (n{-}1, n) \rangle$

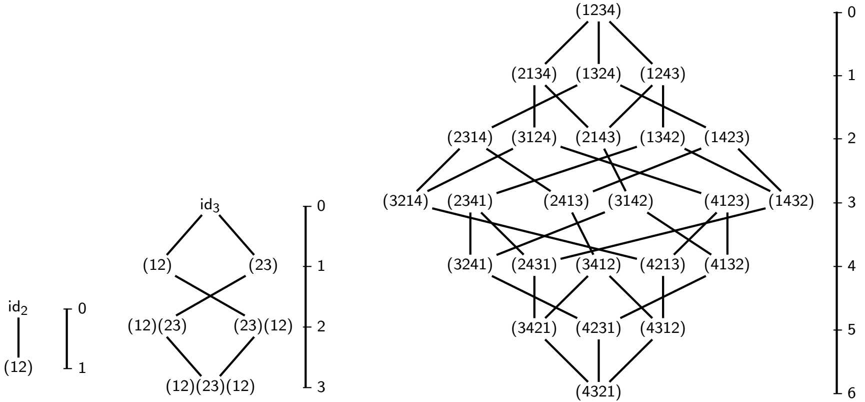
$l(\sigma) := \min \{ m \; ; \; \sigma = \tau_1 \cdots \tau_m \text{ with } \tau_k \in \{(1,2), \dots, (n{-}1, n)\}\}$



Bruhat order on $W(\mathfrak{sl}_n) \cong \mathrm{Sym}(n)$

$$\mathrm{Sym}(n) = \{\text{permutations of } \{1,\dots,n\}\} = \langle (1,2),\dots,(n{-}1,n)\rangle$$

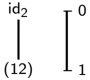
$$l(\sigma) := \min\{m \ ;\ \sigma = \tau_1\cdots\tau_m \text{ with } \tau_k \in \{(1,2),\dots,(n{-}1,n)\}\}$$

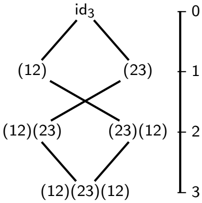


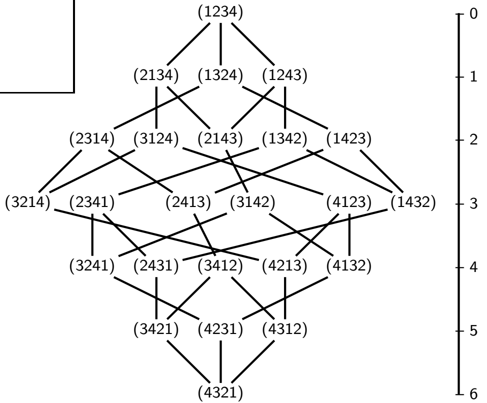
Bruhat order on $W(\mathfrak{sl}_n) \cong \mathrm{Sym}(n)$

$$\mathrm{Sym}(n) = \{\text{permutations of } \{1,\ldots,n\}\} = \langle (1,2),\ldots,(n{-}1,n)\rangle$$

$$l(\sigma) := \min\{m \; ; \; \sigma = \tau_1 \cdots \tau_m \text{ with } \tau_k \in \{(1,2),\ldots,(n{-}1,n)\}\}$$





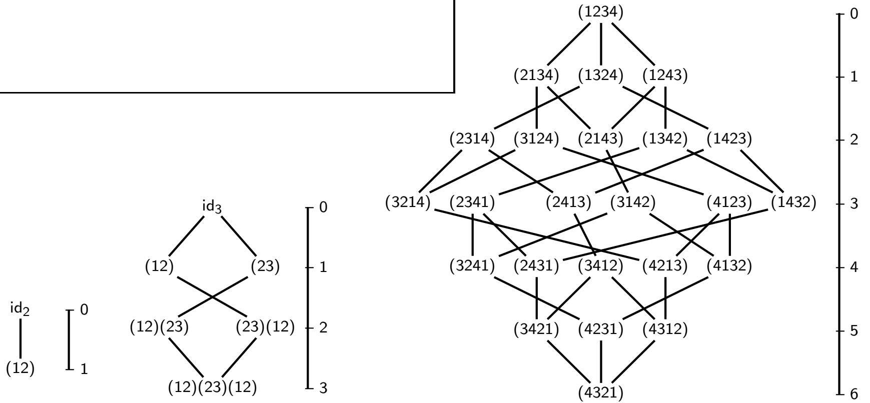


Bruhat order on $W(\mathfrak{sl}_n) \cong \mathrm{Sym}(n)$

$$\mathrm{Sym}(n) = \{\text{permutations of } \{1,\dots,n\}\} = \langle (1,2),\dots,(n-1,n)\rangle$$

$$l(\sigma) := \min\{m \;;\; \sigma = \tau_1 \cdots \tau_m \text{ with } \tau_k \in \{(1,2),\dots,(n-1,n)\}\}$$

$\sigma \lhd \tau$ (σ is a small neighbor of τ)



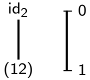
Bruhat order on $W(\mathfrak{sl}_n) \cong \mathrm{Sym}(n)$

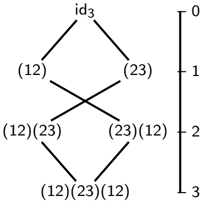
$\mathrm{Sym}(n) = \{\text{permutations of } \{1, \ldots, n\}\} = \langle (1,2), \ldots, (n-1, n) \rangle$

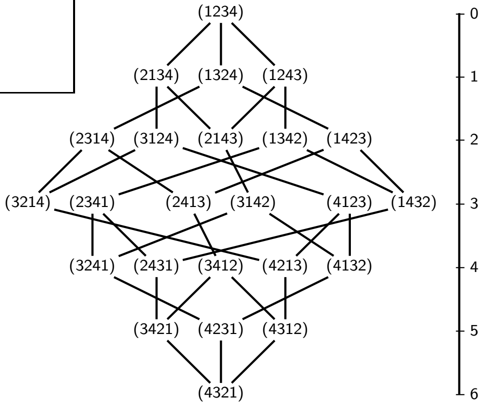
$l(\sigma) := \min \{ m \ ; \ \sigma = \tau_1 \cdots \tau_m \text{ with } \tau_k \in \{(1,2), \ldots, (n-1, n)\}\}$

$\sigma \lhd \tau$ (σ is a small neighbor of τ) if

- $l(\tau) = l(\sigma) - 1$







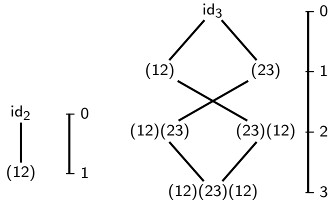
Bruhat order on $W(\mathfrak{sl}_n) \cong \mathrm{Sym}(n)$

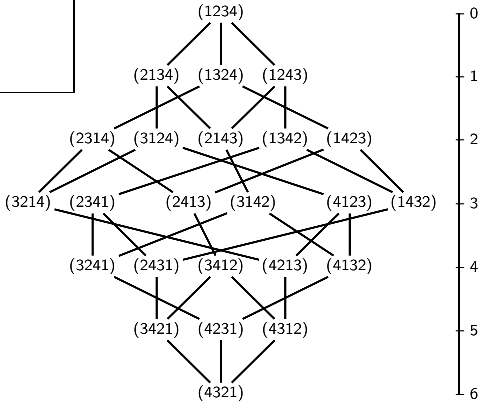
$\mathrm{Sym}(n) = \{\text{permutations of } \{1, \ldots, n\}\} = \langle (1,2), \ldots, (n-1, n) \rangle$

$l(\sigma) := \min \{ m \; ; \; \sigma = \tau_1 \cdots \tau_m \text{ with } \tau_k \in \{(1,2), \ldots, (n-1, n)\}\}$

$\sigma \lhd \tau$ (σ is a small neighbor of τ) if

- $l(\tau) = l(\sigma) - 1$
- $\tau = (i, j) \cdot \sigma$





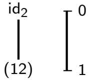
Bruhat order on $W(\mathfrak{sl}_n) \cong \mathrm{Sym}(n)$

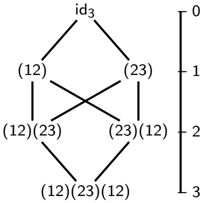
$\mathrm{Sym}(n) = \{\text{permutations of } \{1, \ldots, n\}\} = \langle (1,2), \ldots, (n-1, n) \rangle$

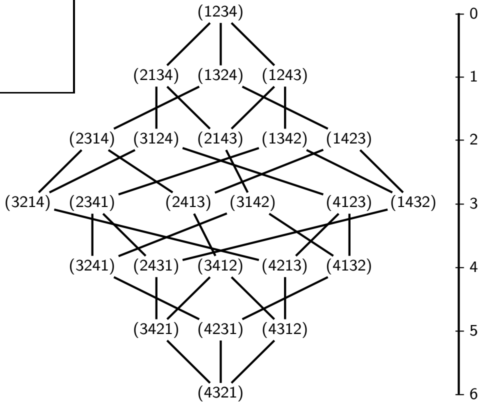
$l(\sigma) := \min\{m \; ; \; \sigma = \tau_1 \cdots \tau_m \text{ with } \tau_k \in \{(1,2), \ldots, (n-1, n)\}\}$

$\sigma \lhd \tau$ (σ is a small neighbor of τ) if

- $l(\tau) = l(\sigma) - 1$
- $\tau = (i, j) \cdot \sigma$







Bruhat order on $W(\mathfrak{sl}_n) \cong \mathrm{Sym}(n)$

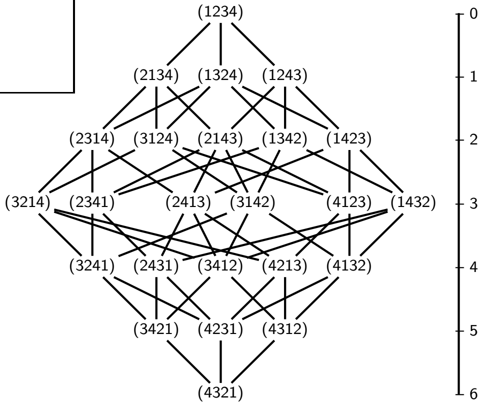
$\mathrm{Sym}(n) = \{\text{permutations of } \{1, \ldots, n\}\} = \langle (1,2), \ldots, (n-1, n) \rangle$

$l(\sigma) := \min \{ m \; ; \; \sigma = \tau_1 \cdots \tau_m \text{ with } \tau_k \in \{(1,2), \ldots, (n-1, n)\}\}$

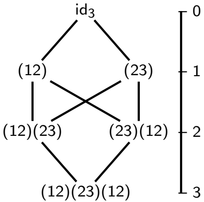
$\sigma \lhd \tau$ (σ is a small neighbor of τ) if

- $l(\tau) = l(\sigma) - 1$
- $\tau = (i, j) \cdot \sigma$

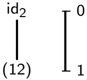
$(\mathrm{Sym}(4), \leqslant)$



$(\mathrm{Sym}(3), \leqslant)$



$(\mathrm{Sym}(2), \leqslant)$



Bruhat order on $W(\mathfrak{sl}_n) \cong \mathrm{Sym}(n)$

$$\mathrm{Sym}(n) = \{\text{permutations of } \{1,\ldots,n\}\} = \langle (1,2),\ldots,(n-1,n)\rangle$$

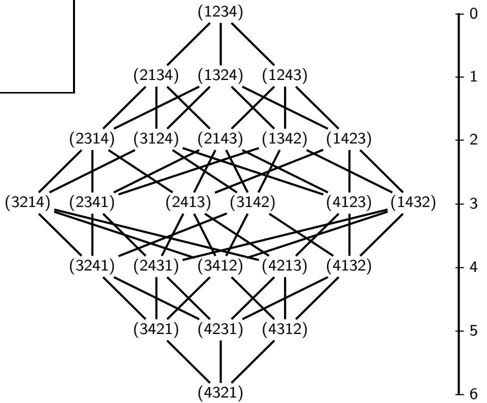
$$l(\sigma) := \min\{m \;;\; \sigma = \tau_1\cdots\tau_m \text{ with } \tau_k \in \{(1,2),\ldots,(n-1,n)\}\}$$

$\sigma \lhd \tau$ (σ is a small neighbor of τ) if

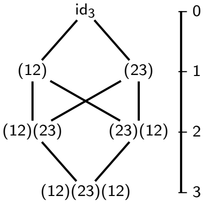
- $l(\tau) = l(\sigma) - 1$
- $\tau = (i,j)\cdot\sigma$

$\sigma < \tau$ if $\sigma \lhd \sigma_1 \lhd \cdots \lhd \sigma_t = \tau$

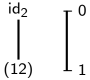
$(\mathrm{Sym}(4), \leqslant)$



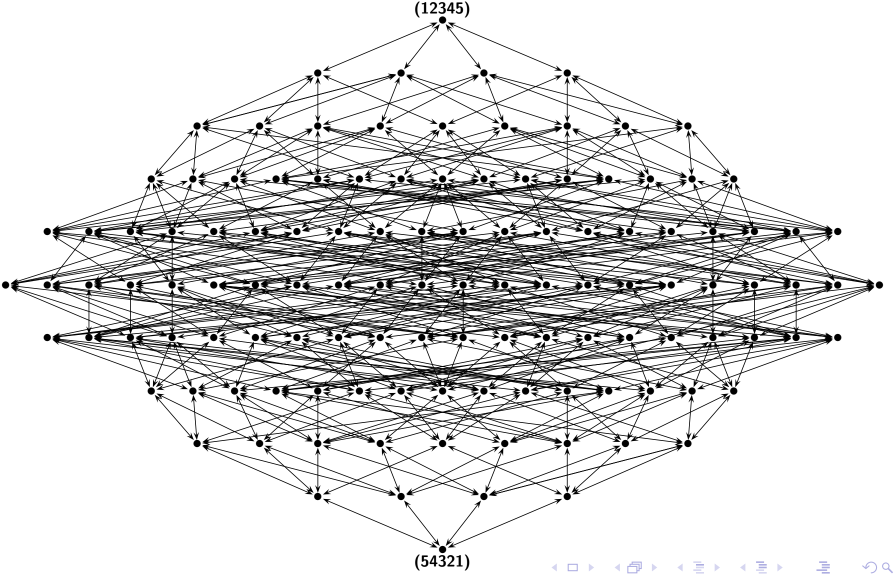
$(\mathrm{Sym}(3), \leqslant)$



$(\mathrm{Sym}(2), \leqslant)$



Bruhat order on $W(\mathfrak{sl}_5) \cong \mathrm{Sym}(5)$



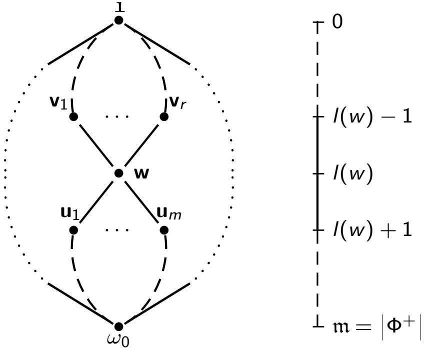
The Hasse-diagram of $(W, \leqslant)$

The Hasse-diagram of $(W, \leqslant)$

$$\boxed{\omega_0 \leqslant w \leqslant \imath \ \text{ for all } w \in W}$$

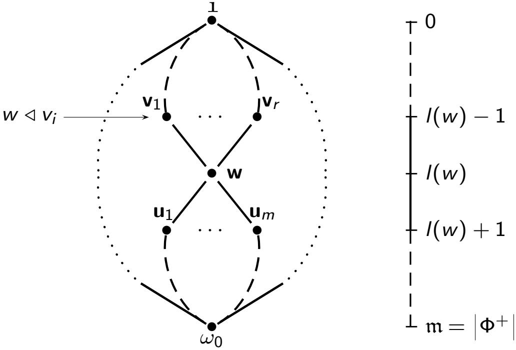
The Hasse-diagram of $(W, \leqslant)$

$$\boxed{\omega_0 \leqslant w \leqslant \imath \ \text{ for all } w \in W}$$



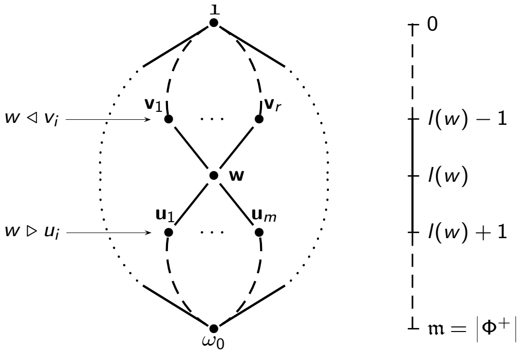
The Hasse-diagram of $(W, \leqslant)$

$$\boxed{\omega_0 \leqslant w \leqslant \imath \ \text{ for all } w \in W}$$



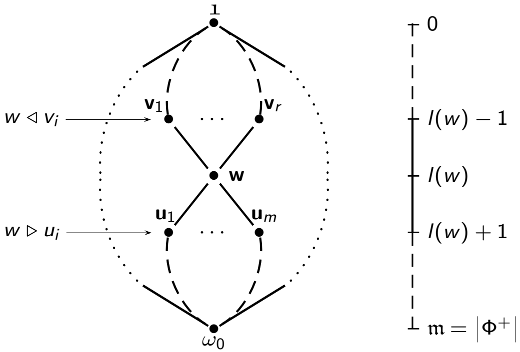
The Hasse-diagram of $(W, \leqslant)$

$$\boxed{\omega_0 \leqslant w \leqslant \imath \ \text{ for all } w \in W}$$



The Hasse-diagram of $(W, \leqslant)$

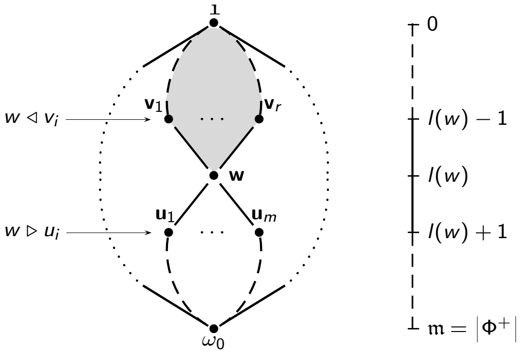
$$\boxed{\omega_0 \leqslant w \leqslant \imath \ \text{ for all } w \in W}$$



- $\Lambda^{(w)} := \{u \in W \ ; \ w \leqslant u\}$

The Hasse-diagram of $(W, \leqslant)$

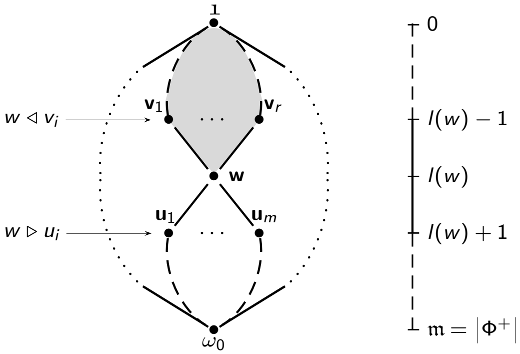
$$\boxed{\omega_0 \leqslant w \leqslant \imath \ \text{ for all } w \in W}$$



- $\Lambda^{(w)} := \{u \in W \ ; \ w \leqslant u\}$

The Hasse-diagram of $(W, \leqslant)$

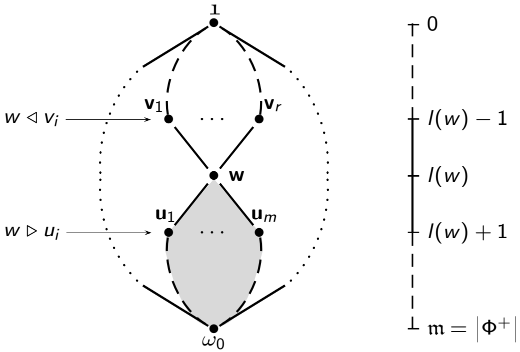
$$\boxed{\omega_0 \leqslant w \leqslant \imath \ \text{ for all } w \in W}$$



- $\Lambda^{(w)} := \{u \in W \ ; \ w \leqslant u\}$
- $\Lambda_{(w)} := \{u \in W \ ; \ w \geqslant u\}$

The Hasse-diagram of $(W, \leqslant)$

$$\boxed{\omega_0 \leqslant w \leqslant \imath \ \text{ for all } w \in W}$$



- $\Lambda^{(w)} := \{u \in W \ ; \ w \leqslant u\}$
- $\Lambda_{(w)} := \{u \in W \ ; \ w \geqslant u\}$

The quiver $Q = (Q_{\mathrm{ver}}, Q_{\mathrm{arr}})$ of algebra $A_0(\mathfrak{g})$

The quiver $Q = (Q_{\mathrm{ver}}, Q_{\mathrm{arr}})$ of algebra $A_0(\mathfrak{g})$

- $Q_{\mathrm{ver}} = W$

The quiver $Q = (Q_{\mathrm{ver}}, Q_{\mathrm{arr}})$ of algebra $A_0(\mathfrak{g})$

- $Q_{\mathrm{ver}} = W$
- $Q_{\mathrm{arr}} \leftrightsquigarrow \mu(w, v) := \left|\left\{\beta \in Q_{\mathrm{arr}} \; ; w \xrightarrow{\beta} v\right\}\right|$

The quiver $Q = (Q_{\text{ver}}, Q_{\text{arr}})$ of algebra $A_0(\mathfrak{g})$

- $Q_{\text{ver}} = W$
- $Q_{\text{arr}} \leftrightsquigarrow \mu(w, v) := \left| \left\{ \beta \in Q_{\text{arr}} \ ; w \xrightarrow{\beta} v \right\} \right| = \dim_{\mathbb{C}} \mathrm{Ext}(S(w), S(v))$

The quiver $Q = (Q_{\mathrm{ver}}, Q_{\mathrm{arr}})$ of algebra $A_0(\mathfrak{g})$

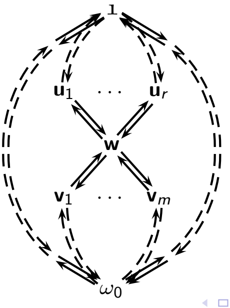
- $Q_{\mathrm{ver}} = W$
- $Q_{\mathrm{arr}} \leftrightsquigarrow \mu(w, v) := \left|\left\{\beta \in Q_{\mathrm{arr}} \;;\, w \xrightarrow{\beta} v\right\}\right| = \dim_{\mathbb{C}} \mathrm{Ext}(S(w), S(v))$
  - $\mu(w, v) = \mu(v, w)$

The quiver $Q = (Q_{\text{ver}}, Q_{\text{arr}})$ of algebra $A_0(\mathfrak{g})$

- $Q_{\text{ver}} = W$
- $Q_{\text{arr}} \leftrightsquigarrow \mu(w,v) := \left|\left\{\beta \in Q_{\text{arr}} \ ; w \xrightarrow{\beta} v\right\}\right| = \dim_{\mathbb{C}} \operatorname{Ext}(S(w), S(v))$
  - $\mu(w,v) = \mu(v,w)$
  - $\mu(w,v) = \begin{cases} 1 & \text{if } w \text{ and } v \text{ are neighbors (Bruhat order)} \\ & \end{cases}$

The quiver $Q = (Q_{\mathrm{ver}}, Q_{\mathrm{arr}})$ of algebra $A_0(\mathfrak{g})$

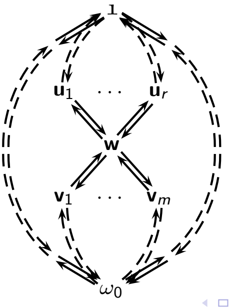
- $Q_{\mathrm{ver}} = W$
- $Q_{\mathrm{arr}} \leftrightsquigarrow \mu(w,v) := \left| \left\{ \beta \in Q_{\mathrm{arr}} \; ; w \xrightarrow{\beta} v \right\} \right| = \dim_{\mathbb{C}} \mathrm{Ext}(S(w), S(v))$
  - $\mu(w,v) = \mu(v,w)$
  - $\mu(w,v) = \begin{cases} 1 & \text{if } w \text{ and } v \text{ are neighbors (Bruhat order)} \\ & \end{cases}$



$\leftrightsquigarrow \quad Q$

The quiver $Q = (Q_{\text{ver}}, Q_{\text{arr}})$ of algebra $A_0(\mathfrak{g})$

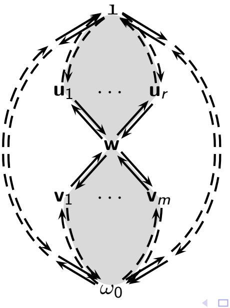
- $Q_{\text{ver}} = W$
- $Q_{\text{arr}} \leftrightsquigarrow \mu(w,v) := \left| \left\{ \beta \in Q_{\text{arr}} \ ; w \xrightarrow{\beta} v \right\} \right| = \dim_{\mathbb{C}} \operatorname{Ext}(S(w), S(v))$
  - $\mu(w,v) = \mu(v,w)$
  - $\mu(w,v) = \begin{cases} 1 & \text{if } w \text{ and } v \text{ are neighbors (Bruhat order)} \\ 0 & \text{if } w \text{ and } v \text{ are incomparable (Bruhat order)} \\ & \end{cases}$



$\leftrightsquigarrow \quad Q$

The quiver $Q = (Q_{\text{ver}}, Q_{\text{arr}})$ of algebra $A_0(\mathfrak{g})$

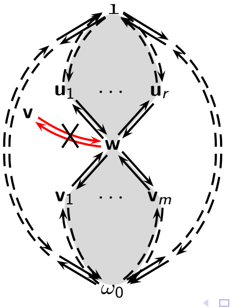
- $Q_{\text{ver}} = W$
- $Q_{\text{arr}} \leftrightsquigarrow \mu(w, v) := \left| \left\{ \beta \in Q_{\text{arr}} \ ; w \xrightarrow{\beta} v \right\} \right| = \dim_{\mathbb{C}} \operatorname{Ext}(S(w), S(v))$
  - $\mu(w, v) = \mu(v, w)$
  - $\mu(w, v) = \begin{cases} 1 & \text{if } w \text{ and } v \text{ are neighbors (Bruhat order)} \\ 0 & \text{if } w \text{ and } v \text{ are incomparable (Bruhat order)} \\ & \end{cases}$



$\leftrightsquigarrow \quad Q$

The quiver $Q = (Q_{\mathrm{ver}}, Q_{\mathrm{arr}})$ of algebra $A_0(\mathfrak{g})$

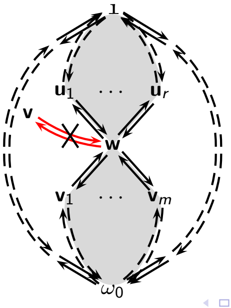
- $Q_{\mathrm{ver}} = W$
- $Q_{\mathrm{arr}} \leftrightsquigarrow \mu(w,v) := \left|\left\{\beta \in Q_{\mathrm{arr}} \ ; w \xrightarrow{\beta} v\right\}\right| = \dim_{\mathbb{C}} \mathrm{Ext}(S(w), S(v))$
  - $\mu(w,v) = \mu(v,w)$
  - $\mu(w,v) = \begin{cases} 1 & \text{if } w \text{ and } v \text{ are neighbors (Bruhat order)} \\ 0 & \text{if } w \text{ and } v \text{ are incomparable (Bruhat order)} \\ & \end{cases}$



$\leftrightsquigarrow \quad Q$

The quiver $Q = (Q_{\mathrm{ver}}, Q_{\mathrm{arr}})$ of algebra $A_0(\mathfrak{g})$

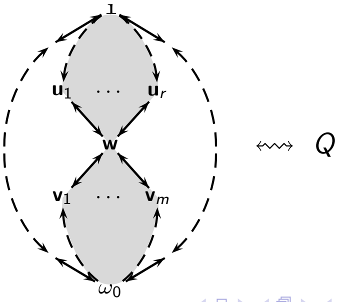
- $Q_{\mathrm{ver}} = W$
- $Q_{\mathrm{arr}} \leftrightsquigarrow \mu(w, v) := \left| \left\{ \beta \in Q_{\mathrm{arr}} \; ; w \xrightarrow{\beta} v \right\} \right| = \dim_{\mathbb{C}} \mathrm{Ext}(S(w), S(v))$
    - $\mu(w, v) = \mu(v, w)$
    - $\mu(w, v) = \begin{cases} 1 & \text{if } w \text{ and } v \text{ are neighbors (Bruhat order)} \\ 0 & \text{if } w \text{ and } v \text{ are incomparable (Bruhat order)} \\ ? & \text{else} \end{cases}$



$\leftrightsquigarrow \quad Q$

The quiver $Q = (Q_{\text{ver}}, Q_{\text{arr}})$ of algebra $A_0(\mathfrak{g})$

- $Q_{\text{ver}} = W$
- $Q_{\text{arr}} \leftrightsquigarrow \mu(w,v) := \left| \left\{ \beta \in Q_{\text{arr}} \; ; w \xrightarrow{\beta} v \right\} \right| = \dim_{\mathbb{C}} \operatorname{Ext}(S(w), S(v))$
  - $\mu(w,v) = \mu(v,w)$
  - $\mu(w,v) = \begin{cases} 1 & \text{if } w \text{ and } v \text{ are neighbors (Bruhat order)} \\ 0 & \text{if } w \text{ and } v \text{ are incomparable (Bruhat order)} \\ ? & \text{else} \end{cases}$



Main Theorem

Main Theorem

Theorem

Main Theorem

Theorem

Let $w \in W$. The following statements are equivalent

Main Theorem

Theorem

Let $w \in W$. The following statements are equivalent

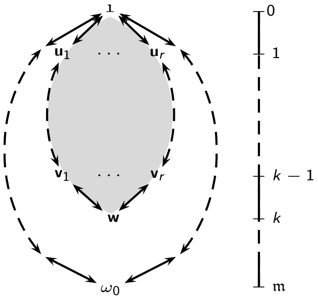
- $\mu(w, v) = 0$ *for all* $v \in \Lambda^{(w)}$ *with* $w \ntriangleleft v$;

Main Theorem

Theorem

Let $w \in W$. The following statements are equivalent

- $\mu(w, v) = 0$ *for all* $v \in \Lambda^{(w)}$ *with* $w \ntriangleleft v$;

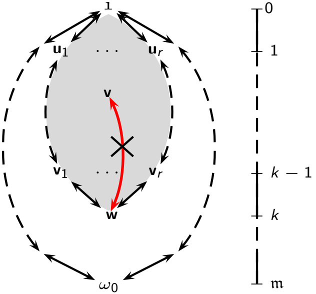


Main Theorem

Theorem

Let $w \in W$. The following statements are equivalent

- $\mu(w, v) = 0$ *for all* $v \in \Lambda^{(w)}$ *with* $w \not\trianglelefteq v$;

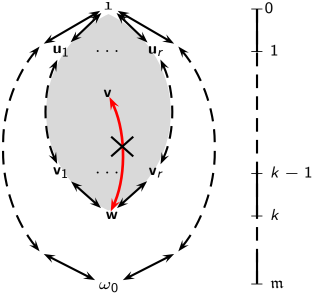


Main Theorem

Theorem

Let $w \in W$. The following statements are equivalent

- $\mu(w, v) = 0$ *for all* $v \in \Lambda^{(w)}$ *with* $w \not\triangleleft v$;
- $\left|\{v \in \Lambda^{(w)}\ ;\ l(v) = i\}\right| = \left|\{v \in \Lambda^{(w)}\ ;\ l(v) = l(w) - i\}\right|\ \forall i$

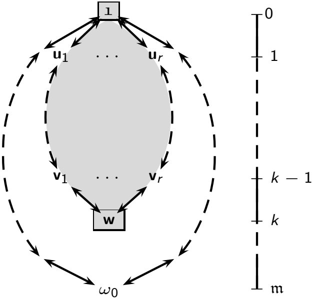


Main Theorem

Theorem

Let $w \in W$. The following statements are equivalent

- $\mu(w, v) = 0$ *for all* $v \in \Lambda^{(w)}$ *with* $w \ntriangleleft v$;
- $\left|\{v \in \Lambda^{(w)}\ ;\ l(v) = i\}\right| = \left|\{v \in \Lambda^{(w)}\ ;\ l(v) = l(w) - i\}\right|\ \forall i$

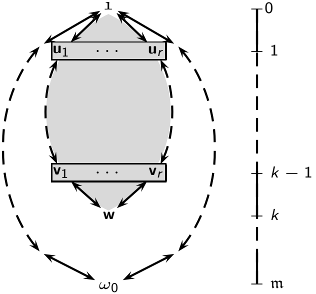


Main Theorem

Theorem

Let $w \in W$. The following statements are equivalent

- $\mu(w,v) = 0$ *for all* $v \in \Lambda^{(w)}$ *with* $w \ntriangleleft v$;
- $\left|\left\{v \in \Lambda^{(w)}\ ;\ l(v) = i\right\}\right| = \left|\left\{v \in \Lambda^{(w)}\ ;\ l(v) = l(w) - i\right\}\right|\ \forall i$

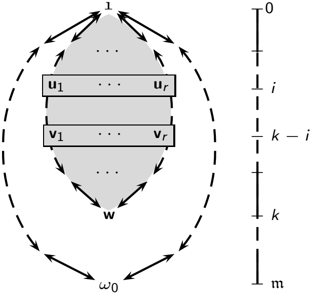


Main Theorem

Theorem

Let $w \in W$. The following statements are equivalent

- $\mu(w, v) = 0$ *for all* $v \in \Lambda^{(w)}$ *with* $w \ntriangleleft v$;
- $\left|\{v \in \Lambda^{(w)}\ ;\ l(v) = i\}\right| = \left|\{v \in \Lambda^{(w)}\ ;\ l(v) = l(w) - i\}\right|\ \forall i$

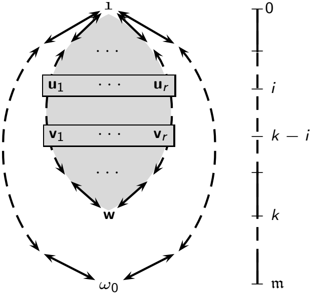


Main Theorem

Theorem

Let $w \in W$. The following statements are equivalent

- $\mu(w, v) = 0$ *for all* $v \in \Lambda^{(w)}$ *with* $w \ntriangleleft v$;
- $\left|\{v \in \Lambda^{(w)}\ ;\ l(v) = i\}\right| = \left|\{v \in \Lambda^{(w)}\ ;\ l(v) = l(w) - i\}\right| \ \forall i$
-
-



Bruhat order on $W(\mathfrak{sl}_n) \cong \mathrm{Sym}(n)$

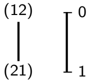
$\mathrm{Sym}(n) = \{\text{permutations of } \{1, \ldots, n\}\} = \langle (1,2), \ldots, (n-1, n) \rangle$

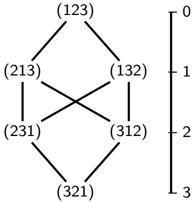
$l(\sigma) := \min\{m \; ; \; \sigma = \tau_1 \cdots \tau_m \text{ with } \tau_k \in \{(1,2), \ldots, (n-1, n)\}\}$

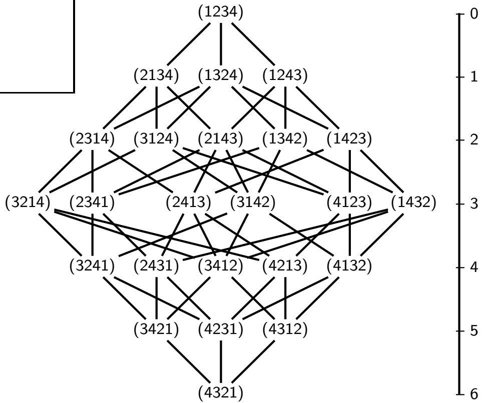
$\sigma \lhd \tau$ (σ is a small neighbor of τ) if

- $l(\tau) = l(\sigma) - 1$
- $\tau = (i, j) \cdot \sigma$

$\sigma < \tau$ if $\sigma \lhd \sigma_1 \lhd \cdots \lhd \sigma_t = \tau$







Bruhat order on $W(\mathfrak{sl}_n) \cong \mathrm{Sym}(n)$

$\mathrm{Sym}(n) = \{\text{permutations of } \{1,\dots,n\}\} = \langle (1,2),\dots,(n-1,n)\rangle$

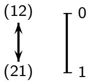
$l(\sigma) := \min\{m \; ; \; \sigma = \tau_1 \cdots \tau_m \text{ with } \tau_k \in \{(1,2),\dots,(n-1,n)\}\}$

$\sigma \lhd \tau$ (σ is a small neighbor of τ) if

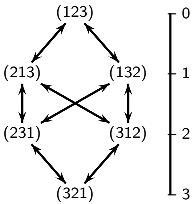
- $l(\tau) = l(\sigma) - 1$
- $\tau = (i,j) \cdot \sigma$

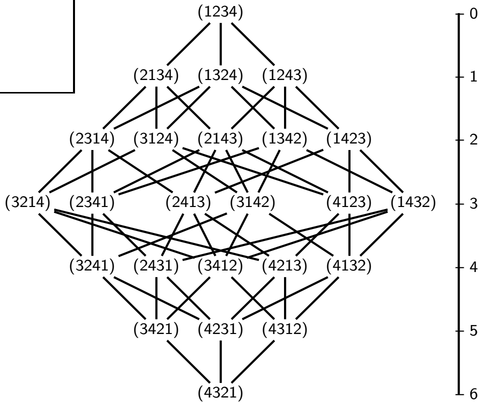
$\sigma < \tau$ if $\sigma \lhd \sigma_1 \lhd \cdots \lhd \sigma_t = \tau$

$A_0(\mathfrak{sl}_2)$



$A_0(\mathfrak{sl}_3)$





Bruhat order on $W(\mathfrak{sl}_n) \cong \mathrm{Sym}(n)$

$\mathrm{Sym}(n) = \{\text{permutations of } \{1, \ldots, n\}\} = \langle (1,2), \ldots, (n{-}1, n) \rangle$

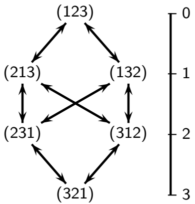
$l(\sigma) := \min \{m \; ; \; \sigma = \tau_1 \cdots \tau_m \text{ with } \tau_k \in \{(1,2), \ldots, (n{-}1, n)\}\}$

$\sigma \lhd \tau$ (σ is a small neighbor of τ) if

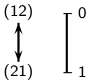
- $l(\tau) = l(\sigma) - 1$
- $\tau = (i,j) \cdot \sigma$

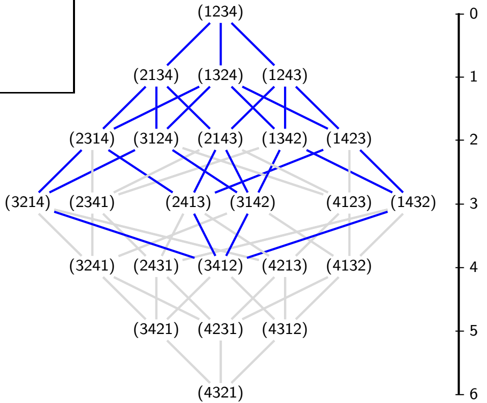
$\sigma < \tau$ if $\sigma \lhd \sigma_1 \lhd \cdots \lhd \sigma_t = \tau$

$A_0(\mathfrak{sl}_3)$



$A_0(\mathfrak{sl}_2)$





Bruhat order on $W(\mathfrak{sl}_n) \cong \mathrm{Sym}(n)$

$\mathrm{Sym}(n) = \{\text{permutations of } \{1, \dots, n\}\} = \langle (1,2), \dots, (n-1, n) \rangle$

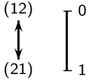
$l(\sigma) := \min\{m \; ; \; \sigma = \tau_1 \cdots \tau_m \text{ with } \tau_k \in \{(1,2), \dots, (n-1,n)\}\}$

$\sigma \lhd \tau$ (σ is a small neighbor of τ) if

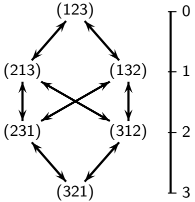
- $l(\tau) = l(\sigma) - 1$
- $\tau = (i, j) \cdot \sigma$

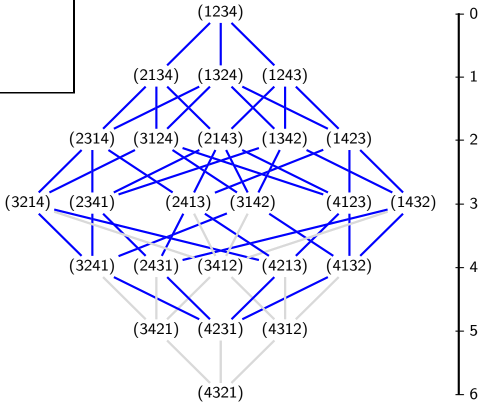
$\sigma < \tau$ if $\sigma \lhd \sigma_1 \lhd \cdots \lhd \sigma_t = \tau$

$A_0(\mathfrak{sl}_2)$



$A_0(\mathfrak{sl}_3)$





Bruhat order on $W(\mathfrak{sl}_n) \cong \mathrm{Sym}(n)$

$\mathrm{Sym}(n) = \{\text{permutations of } \{1,\dots,n\}\} = \langle (1,2),\dots,(n-1,n)\rangle$

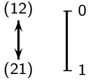
$l(\sigma) := \min\{m \,;\, \sigma = \tau_1\cdots\tau_m \text{ with } \tau_k \in \{(1,2),\dots,(n-1,n)\}\}$

$\sigma \lhd \tau$ (σ is a small neighbor of τ) if

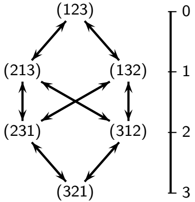
- $l(\tau) = l(\sigma) - 1$
- $\tau = (i,j)\cdot\sigma$

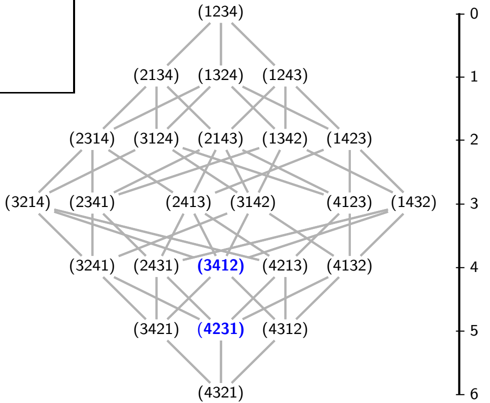
$\sigma < \tau$ if $\sigma \lhd \sigma_1 \lhd \cdots \lhd \sigma_t = \tau$

$A_0(\mathfrak{sl}_2)$



$A_0(\mathfrak{sl}_3)$





Bruhat order on $W(\mathfrak{sl}_n) \cong \mathrm{Sym}(n)$

$\mathrm{Sym}(n) = \{\text{permutations of } \{1, \ldots, n\}\} = \langle (1,2), \ldots, (n-1, n) \rangle$

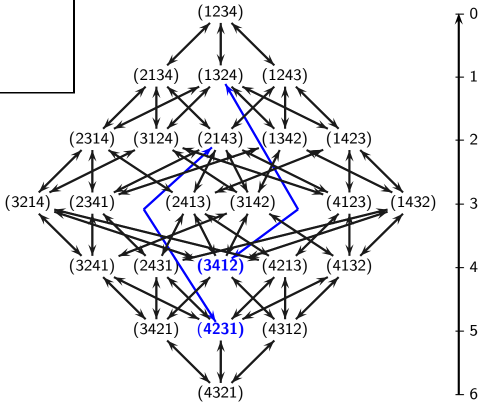
$l(\sigma) := \min \{ m \; ; \; \sigma = \tau_1 \cdots \tau_m \text{ with } \tau_k \in \{(1,2), \ldots, (n-1,n)\}\}$

$\sigma \lhd \tau$ (σ is a small neighbor of τ) if

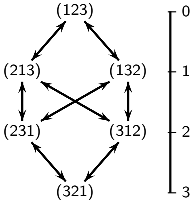
- $l(\tau) = l(\sigma) - 1$
- $\tau = (i,j) \cdot \sigma$

$\sigma < \tau$ if $\sigma \lhd \sigma_1 \lhd \cdots \lhd \sigma_t = \tau$

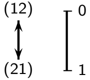
$A_0(\mathfrak{sl}_4)$



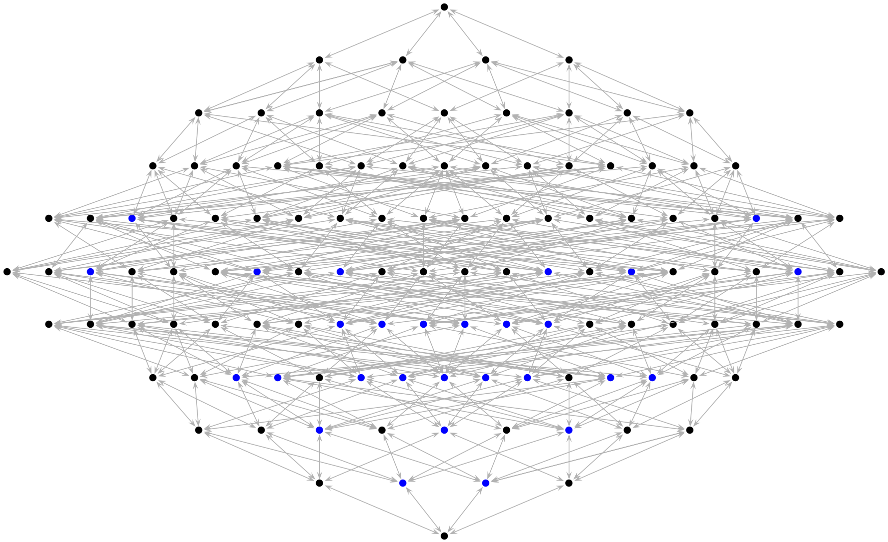
$A_0(\mathfrak{sl}_3)$



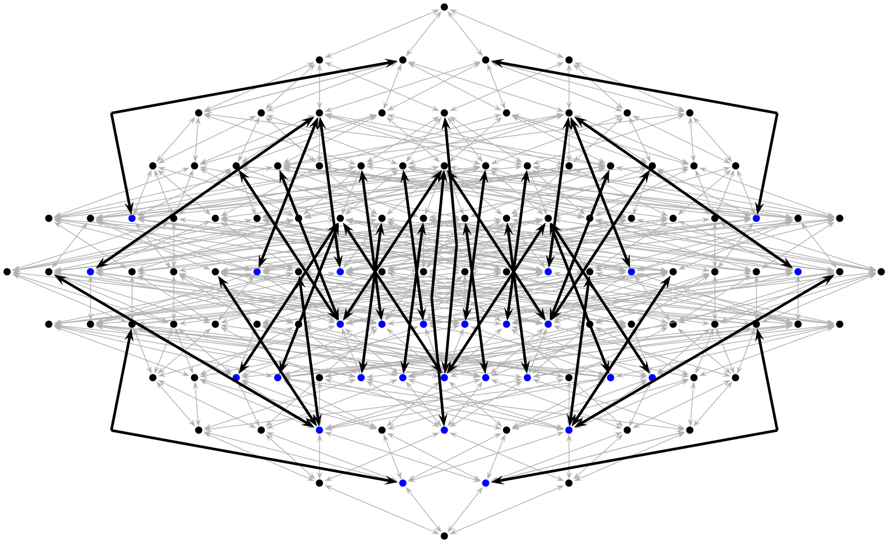
$A_0(\mathfrak{sl}_2)$



Bruhat order on $W(\mathfrak{sl}_5) \cong \mathrm{Sym}(5)$



Bruhat order on $W(\mathfrak{sl}_5) \cong \mathrm{Sym}(5)$



Connection to quasi-hereditary algebras

Connection to quasi-hereditary algebras

$$\boxed{A_0(\mathfrak{g}) \text{ with } (W, \leqslant) \text{ is quasi-hereditary}}$$

$$0 = M_0 \subset \cdots \subset \underbrace{M_{i-1} \subset M_i}_{\Delta(v)} \subset \cdots \subset \underbrace{M_{t-1} \subset M_t}_{\Delta(w)} = P(w)$$

Connection to quasi-hereditary algebras

$$\boxed{A_0(\mathfrak{g}) \text{ with } (W, \leqslant) \text{ is quasi-hereditary}}$$

$$0 = M_0 \subset \cdots \subset \underbrace{M_{i-1} \subset M_i}_{\Delta(v)} \subset \cdots \subset \underbrace{M_{t-1} \subset M_t}_{\Delta(w)} = P(w)$$

$$(P(w) : \Delta(v)) := |\{i \ ; \ M_i / M_{i-1} \cong \Delta(v)\}|$$

Connection to quasi-hereditary algebras

$$\boxed{A_0(\mathfrak{g}) \text{ with } (W, \leqslant) \text{ is quasi-hereditary}}$$

$$0 = M_0 \subset \cdots \subset \underbrace{M_{i-1} \subset M_i}_{\Delta(v)} \subset \cdots \subset \underbrace{M_{t-1} \subset M_t}_{\Delta(w)} = P(w)$$

$(P(w) : \Delta(v)) := |\{i \; ; \; M_i/M_{i-1} \cong \Delta(v)\}|$ is well defined

Connection to quasi-hereditary algebras

$$\boxed{A_0(\mathfrak{g}) \text{ with } (W, \leqslant) \text{ is quasi-hereditary}}$$

$$0 = M_0 \subset \cdots \subset \underbrace{M_{i-1} \subset M_i}_{\Delta(v)} \subset \cdots \subset \underbrace{M_{t-1} \subset M_t}_{\Delta(w)} = P(w)$$

$(P(w) : \Delta(v)) := |\{i \ ; \ M_i/M_{i-1} \cong \Delta(v)\}|$ is well defined

- $(P(w) : \Delta(w)) = 1$

Connection to quasi-hereditary algebras

$$\boxed{A_0(\mathfrak{g}) \text{ with } (W, \leqslant) \text{ is quasi-hereditary}}$$

$$0 = M_0 \subset \cdots \subset \underbrace{M_{i-1} \subset M_i}_{\Delta(v)} \subset \cdots \subset \underbrace{M_{t-1} \subset M_t}_{\Delta(w)} = P(w)$$

$$(P(w) : \Delta(v)) := |\{i \ ; \ M_i/M_{i-1} \cong \Delta(v)\}| \text{ is well defined}$$

- $(P(w) : \Delta(w)) = 1$
- $(P(w) : \Delta(v)) \neq 0$

Connection to quasi-hereditary algebras

$$\boxed{A_0(\mathfrak{g}) \text{ with } (W, \leqslant) \text{ is quasi-hereditary}}$$

$$0 = M_0 \subset \cdots \subset \underbrace{M_{i-1} \subset M_i}_{\Delta(v)} \subset \cdots \subset \underbrace{M_{t-1} \subset M_t}_{\Delta(w)} = P(w)$$

$(P(w) : \Delta(v)) := |\{i \ ; \ M_i/M_{i-1} \cong \Delta(v)\}|$ is well defined

- $(P(w) : \Delta(w)) = 1$
- $(P(w) : \Delta(v)) \neq 0 \ \Rightarrow \ v \in \Lambda^{(w)}$

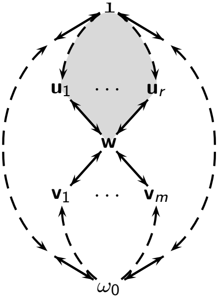
Connection to quasi-hereditary algebras

$$\boxed{A_0(\mathfrak{g}) \text{ with } (W, \leqslant) \text{ is quasi-hereditary}}$$

$$0 = M_0 \subset \cdots \subset \underbrace{M_{i-1} \subset M_i}_{\Delta(v)} \subset \cdots \subset \underbrace{M_{t-1} \subset M_t}_{\Delta(w)} = P(w)$$

$(P(w) : \Delta(v)) := |\{i \ ; \ M_i/M_{i-1} \cong \Delta(v)\}|$ is well defined

- $(P(w) : \Delta(w)) = 1$
- $(P(w) : \Delta(v)) \neq 0 \ \Rightarrow \ v \in \Lambda^{(w)}$



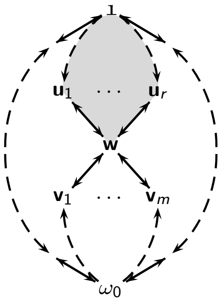
Connection to quasi-hereditary algebras

$$\boxed{A_0(\mathfrak{g}) \text{ with } (W, \leqslant) \text{ is quasi-hereditary}}$$

$$0 = M_0 \subset \cdots \subset \underbrace{M_{i-1} \subset M_i}_{\Delta(v)} \subset \cdots \subset \underbrace{M_{t-1} \subset M_t}_{\Delta(w)} = P(w)$$

$(P(w) : \Delta(v)) := |\{i\ ;\ M_i/M_{i-1} \cong \Delta(v)\}|$ is well defined

- $(P(w) : \Delta(w)) = 1$
- $(P(w) : \Delta(v)) \neq 0 \ \Leftrightarrow\ v \in \Lambda^{(w)}$



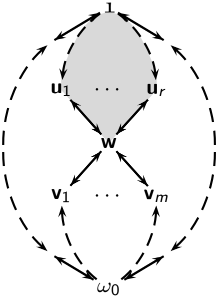
Connection to quasi-hereditary algebras

$\boxed{A_0(\mathfrak{g}) \text{ with } (W, \leqslant) \text{ is quasi-hereditary}}$

$$0 = M_0 \subset \cdots \subset \underbrace{M_{i-1} \subset M_i}_{\Delta(v)} \subset \cdots \subset \underbrace{M_{t-1} \subset M_t}_{\Delta(w)} = P(w)$$

$(P(w) : \Delta(v)) := |\{i \; ; \; M_i/M_{i-1} \cong \Delta(v)\}|$ is well defined

- $(P(w) : \Delta(w)) = 1$
- $(P(w) : \Delta(v)) \neq 0 \;\; \Leftrightarrow \;\; v \in \Lambda^{(w)}$

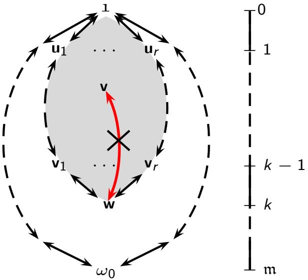


Main Theorem

Theorem

Let $w \in W$. The following statements are equivalent

- $\mu(w,v) = 0$ *for all* $v \in \Lambda^{(w)}$ *with* $w \ntriangleleft v$;
- $\left|\left\{v \in \Lambda^{(w)} \;;\; l(v) = i\right\}\right| = \left|\left\{v \in \Lambda^{(w)} \;;\; l(v) = l(w) - i\right\}\right| \;\forall i$

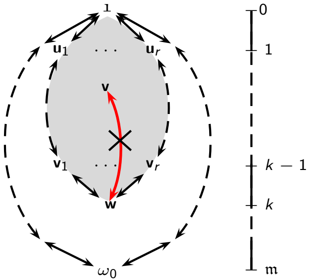


Main Theorem

Theorem

Let $w \in W$. The following statements are equivalent

- $\mu(w, v) = 0$ *for all* $v \in \Lambda^{(w)}$ *with* $w \ntriangleleft v$;
- $\left|\left\{v \in \Lambda^{(w)} \ ; \ l(v) = i\right\}\right| = \left|\left\{v \in \Lambda^{(w)} \ ; \ l(v) = l(w) - i\right\}\right| \ \forall i$
- $(P(w) : \Delta(v)) = 1$ *for all* $v \in \Lambda^{(w)}$

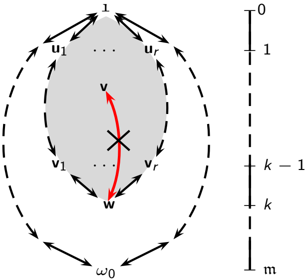


Main Theorem

Theorem

Let $w \in W$. The following statements are equivalent

- $\mu(w, v) = 0$ *for all* $v \in \Lambda^{(w)}$ *with* $w \not\trianglelefteq v$;
- $\left|\{v \in \Lambda^{(w)} \; ; \; l(v) = i\}\right| = \left|\{v \in \Lambda^{(w)} \; ; \; l(v) = l(w) - i\}\right| \; \forall i$
- $(P(w) : \Delta(v)) = 1$ *for all* $v \in \Lambda^{(w)}$
- $\operatorname{soc} P(w)$ *is simple*



Connection to Ringel dual of $A := A_0(\mathfrak{g})$

Connection to Ringel dual of $A := A_0(\mathfrak{g})$

- $A(w) := A/A\left(\sum_{v \notin \Lambda_{(w)}} e_v\right) A$

Connection to Ringel dual of $A := A_0(\mathfrak{g})$

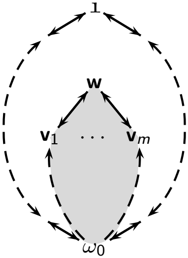
- $A(w) := A/A\left(\sum_{v \notin \Lambda_{(w)}} e_v\right)A \cong \mathbb{C}Q(w)/\mathcal{I}(w)$

Connection to Ringel dual of $A := A_0(\mathfrak{g})$

- $A(w) := A/A\left(\sum_{v \notin \Lambda_{(w)}} e_v\right)A \cong \mathbb{C}Q(w)/\mathcal{I}(w)$
    - $Q_{\text{ver}}(w) = \Lambda_{(w)}$

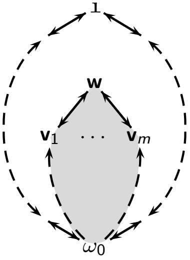
Connection to Ringel dual of $A := A_0(\mathfrak{g})$

- $A(w) := A/A\left(\sum_{v \notin \Lambda_{(w)}} e_v\right) A \cong \mathbb{C}Q(w)/\mathcal{I}(w)$
  - $Q_{\text{ver}}(w) = \Lambda_{(w)}$



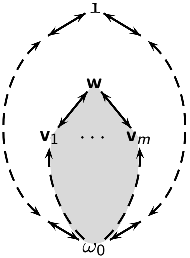
Connection to Ringel dual of $A := A_0(\mathfrak{g})$

- $A(w) := A/A\left(\sum_{v \notin \Lambda_{(w)}} e_v\right) A \cong \mathbb{C}Q(w)/\mathcal{I}(w)$
  - $Q_{\mathrm{ver}}(w) = \Lambda_{(w)}$
  - $Q_{\mathrm{arr}}(w) = \left\{ v \to u \in Q_{\mathrm{arr}} \ ; \ v, u \in \Lambda_{(w)} \right\}$



Connection to Ringel dual of $A := A_0(\mathfrak{g})$

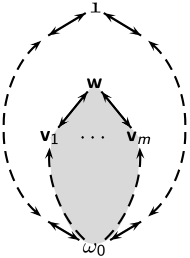
- $A(w) := A/A\left(\sum_{v \notin \Lambda_{(w)}} e_v\right) A \cong \mathbb{C}Q(w)/\mathcal{I}(w)$
  - $Q_{\mathrm{ver}}(w) = \Lambda_{(w)}$
  - $Q_{\mathrm{arr}}(w) = \left\{ v \to u \in Q_{\mathrm{arr}} \ ; \ v, u \in \Lambda_{(w)} \right\}$



- $\operatorname{mod} A(w) = \{ M \in \operatorname{mod} A \ ; \ [M : S(u)] = 0, \forall \ u \ \not\leq w \}$

Connection to Ringel dual of $A := A_0(\mathfrak{g})$

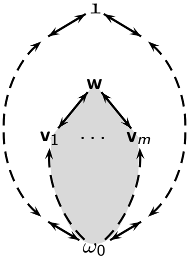
- $A(w) := A/A\left(\sum_{v \notin \Lambda_{(w)}} e_v\right) A \cong \mathbb{C}Q(w)/\mathcal{I}(w)$
  - $Q_{\mathrm{ver}}(w) = \Lambda_{(w)}$
  - $Q_{\mathrm{arr}}(w) = \left\{v \to u \in Q_{\mathrm{arr}} \ ; \ v, u \in \Lambda_{(w)}\right\}$



- $\mathrm{mod}\, A(w) = \{M \in \mathrm{mod}\, A \ ; \ [M : S(u)] = 0, \forall \ u \ \not\leq w\}$

Connection to Ringel dual of $A := A_0(\mathfrak{g})$

- $A(w) := A/A\left(\sum_{v \notin \Lambda_{(w)}} e_v\right) A \cong \mathbb{C}Q(w)/\mathcal{I}(w)$
  - $Q_{\mathrm{ver}}(w) = \Lambda_{(w)}$
  - $Q_{\mathrm{arr}}(w) = \left\{ v \to u \in Q_{\mathrm{arr}} \ ; \ v, u \in \Lambda_{(w)} \right\}$

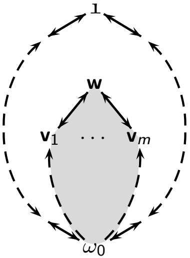


- $\mathrm{mod}\, A(w) = \{ M \in \mathrm{mod}\, A \ ; \ [M : S(u)] = 0, \forall\ u \not\leq w \}$

For any $w \in W$ there is an unique A-module $T(w)$ with

Connection to Ringel dual of $A := A_0(\mathfrak{g})$

- $A(w) := A/A\left(\sum_{v \notin \Lambda_{(w)}} e_v\right) A \cong \mathbb{C}Q(w)/\mathcal{I}(w)$
  - $Q_{\text{ver}}(w) = \Lambda_{(w)}$
  - $Q_{\text{arr}}(w) = \left\{v \to u \in Q_{\text{arr}} \; ; \; v, u \in \Lambda_{(w)}\right\}$

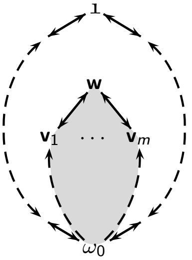


- $\operatorname{mod} A(w) = \{M \in \operatorname{mod} A \; ; \; [M : S(u)] = 0, \forall \; u \not\leq w\}$

For any $w \in W$ there is an unique A-module $T(w)$ with
- $T(w)$ is an indecomposable $A(w)$-module

Connection to Ringel dual of $A := A_0(\mathfrak{g})$

- $A(w) := A/A\left(\sum_{v \notin \Lambda_{(w)}} e_v\right)A \cong \mathbb{C}Q(w)/\mathcal{I}(w)$
  - $Q_{\mathrm{ver}}(w) = \Lambda_{(w)}$
  - $Q_{\mathrm{arr}}(w) = \left\{v \to u \in Q_{\mathrm{arr}} \ ; \ v, u \in \Lambda_{(w)}\right\}$

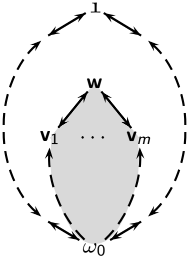


- $\mathrm{mod}\, A(w) = \{M \in \mathrm{mod}\, A \ ; \ [M : S(u)] = 0, \forall\ u \not\leq w\}$

For any $w \in W$ there is an unique A-module $T(w)$ with
- $T(w)$ is an indecomposable $A(w)$-module
- $T(w)$ has a Δ- and a ∇-filtration

Connection to Ringel dual of $A := A_0(\mathfrak{g})$

- $A(w) := A/A\left(\sum_{v \notin \Lambda_{(w)}} e_v\right) A \cong \mathbb{C}Q(w)/\mathcal{I}(w)$
  - $Q_{ver}(w) = \Lambda_{(w)}$
  - $Q_{arr}(w) = \left\{v \to u \in Q_{arr} \ ; \ v, u \in \Lambda_{(w)}\right\}$



- $\operatorname{mod} A(w) = \{M \in \operatorname{mod} A \ ; \ [M : S(u)] = 0, \forall \ u \not\leq w\}$

For any $w \in W$ there is an unique A-module $T(w)$ with
- $T(w)$ is an indecomposable $A(w)$-module
- $T(w)$ has a Δ- and a ∇-filtration
- $[T(w) : S(w)] = (T(w) : \Delta(w)) = (T(w) : \nabla(w)) = 1$

Connection to Ringel dual of $A := A_0(\mathfrak{g})$

- Important modules in $\mathcal{O}_0(\mathfrak{g})$

$$
\begin{array}{ccll}
 & \nearrow\!\!\!\swarrow & \{\text{ Verma module } \Delta(w)\ \} \longleftrightarrow & \{\text{coVerma module } \nabla(w)\} \\
w \in W & \longleftrightarrow & \{\text{ simple module } S(w)\ \} & \\
 & \searrow\!\!\!\nwarrow & \{\text{projective indec. } P(w)\ \} \longleftrightarrow & \{\text{injective indec. } I(w)\}
\end{array}
$$

Connection to Ringel dual of $A := A_0(\mathfrak{g})$

- Important modules in $\mathcal{O}_0(\mathfrak{g})$

$$
\begin{tikzcd}
 & \{\text{ Verma module } \Delta(w)\ \} \arrow[r, leftrightarrow] & \{\text{coVerma module } \nabla(w)\} \\
w \in W \arrow[ur, leftrightarrow] \arrow[r, leftrightarrow] \arrow[dr, leftrightarrow] \arrow[ddr, leftrightarrow] & \{\text{ simple module } S(w)\ \} & \\
 & \{\text{projective indec. } P(w)\ \} \arrow[r, leftrightarrow] & \{\text{injective indec. } I(w)\} \\
 & \{\text{tilting indec. } T(w)\ \} &
\end{tikzcd}
$$

Connection to Ringel dual of $A := A_0(\mathfrak{g})$

- Important modules in $\mathcal{O}_0(\mathfrak{g})$

$$\begin{tikzcd}
 & \{\text{ Verma module } \Delta(w)\ \} \arrow[r, leftrightarrow] & \{\text{coVerma module } \nabla(w)\} \\
w \in W \arrow[ur, leftrightarrow] \arrow[r, leftrightarrow] \arrow[dr, leftrightarrow] \arrow[ddr, leftrightarrow] & \{\text{ simple module } S(w)\ \} & \\
 & \{\text{projective indec. } P(w)\ \} \arrow[r, leftrightarrow] & \{\text{injective indec. } I(w)\} \\
 & \{\text{tilting indec. } T(w)\ \} &
\end{tikzcd}$$

- $R(A) := \mathrm{End}_{\mathfrak{g}} \left(\bigoplus_{w \in W} T(w) \right)^{op}$

Connection to Ringel dual of $A := A_0(\mathfrak{g})$

- Important modules in $\mathcal{O}_0(\mathfrak{g})$

$$
\begin{tikzcd}[column sep=small]
 & \{\text{ Verma module } \Delta(w)\ \} \arrow[r, leftrightarrow] & \{\text{coVerma module } \nabla(w)\} \\
w \in W \arrow[ur, leftrightarrow] \arrow[r, leftrightarrow] \arrow[dr, leftrightarrow] \arrow[ddr, leftrightarrow] & \{\text{ simple module } S(w)\ \} & \\
 & \{\text{projective indec. } P(w)\ \} \arrow[r, leftrightarrow] & \{\text{injective indec. } I(w)\} \\
 & \{\text{tilting indec. } T(w)\ \} &
\end{tikzcd}
$$

- $R(A) := \mathrm{End}_{\mathfrak{g}}\left(\bigoplus_{w\in W} T(w)\right)^{op}$ is Ringel dual

Connection to Ringel dual of $A := A_0(\mathfrak{g})$

- Important modules in $\mathcal{O}_0(\mathfrak{g})$

$$
\begin{tikzcd}
 & \{\text{ Verma module } \Delta(w)\ \} \arrow[r, leftrightarrow] & \{\text{coVerma module } \nabla(w)\} \\
w \in W \arrow[ur, leftrightarrow] \arrow[r, leftrightarrow] \arrow[dr, leftrightarrow] \arrow[ddr, leftrightarrow] & \{\text{ simple module } S(w)\ \} & \\
 & \{\text{projective indec. } P(w)\ \} \arrow[r, leftrightarrow] & \{\text{injective indec. } I(w)\} \\
 & \{\text{tilting indec. } T(w)\ \} &
\end{tikzcd}
$$

- $R(A) := \mathrm{End}_{\mathfrak{g}}\left(\bigoplus_{w \in W} T(w)\right)^{op}$ is Ringel dual

$$\boxed{(R(A), \geqslant) \text{ is quasi-hereditary}}$$

Connection to Ringel dual of $A := A_0(\mathfrak{g})$

- Important modules in $\mathcal{O}_0(\mathfrak{g})$

$$\begin{tikzcd}[row sep=small]
 & \{\text{ Verma module } \Delta(w)\ \} \arrow[r, leftrightarrow] & \{\text{coVerma module } \nabla(w)\} \\
w \in W \arrow[ur, leftrightarrow] \arrow[r, leftrightarrow] \arrow[dr, leftrightarrow] \arrow[ddr, leftrightarrow] & \{\text{ simple module } S(w)\ \} & \\
 & \{\text{projective indec. } P(w)\ \} \arrow[r, leftrightarrow] & \{\text{injective indec. } I(w)\} \\
 & \{\text{tilting indec. } T(w)\ \} &
\end{tikzcd}$$

- $R(A) := \mathrm{End}_{\mathfrak{g}}\left(\bigoplus_{w\in W} T(w)\right)^{op}$ is Ringel dual

$$\boxed{(R(A), \geqslant) \text{ is quasi-hereditary}}$$

- $\mathrm{Hom}_{\mathfrak{g}}\left(\bigoplus_{w\in W} T(w), -\right) : \mathrm{mod}A \to \mathrm{mod}R(A)$ is exact on $\mathcal{F}(\nabla)$

Connection to Ringel dual of $A := A_0(\mathfrak{g})$

- Important modules in $\mathcal{O}_0(\mathfrak{g})$

$$
\begin{tikzcd}[row sep=small]
 & \{\text{ Verma module } \Delta(w)\ \} \arrow[r, leftrightarrow] & \{\text{coVerma module } \nabla(w)\} \\
w \in W \arrow[ur, leftrightarrow] \arrow[r, leftrightarrow] \arrow[dr, leftrightarrow] \arrow[ddr, leftrightarrow] & \{\text{ simple module } S(w)\ \} & \\
 & \{\text{projective indec. } P(w)\ \} \arrow[r, leftrightarrow] & \{\text{injective indec. } I(w)\} \\
 & \{\text{tilting indec. } T(w)\ \} &
\end{tikzcd}
$$

- $R(A) := \mathrm{End}_{\mathfrak{g}}\left(\bigoplus_{w\in W} T(w)\right)^{op}$ is Ringel dual

$$\boxed{(R(A), \geqslant) \text{ is quasi-hereditary}}$$

- $\mathrm{Hom}_{\mathfrak{g}}\left(\bigoplus_{w\in W} T(w), -\right) : \mathrm{mod} A \to \mathrm{mod} R(A)$ is exact on $\mathcal{F}(\nabla)$

$$\nabla(w) \quad \mapsto \quad \Delta_R(w)$$

Connection to Ringel dual of $A := A_0(\mathfrak{g})$

- Important modules in $\mathcal{O}_0(\mathfrak{g})$

$$
\begin{tikzcd}[column sep=small]
 & \{\text{ Verma module } \Delta(w)\ \} \arrow[r, leftrightarrow] & \{\text{coVerma module } \nabla(w)\} \\
w \in W \arrow[ur, leftrightarrow] \arrow[r, leftrightarrow] \arrow[dr, leftrightarrow] \arrow[ddr, leftrightarrow] & \{\text{ simple module } S(w)\ \} & \\
 & \{\text{projective indec. } P(w)\ \} \arrow[r, leftrightarrow] & \{\text{injective indec. } I(w)\} \\
 & \{\text{tilting indec. } T(w)\ \} &
\end{tikzcd}
$$

- $R(A) := \mathrm{End}_{\mathfrak{g}}\left(\bigoplus_{w \in W} T(w)\right)^{op}$ is Ringel dual

$$\boxed{(R(A), \geqslant) \text{ is quasi-hereditary}}$$

- $\mathrm{Hom}_{\mathfrak{g}}\left(\bigoplus_{w \in W} T(w), -\right) : \mathrm{mod} A \to \mathrm{mod} R(A)$ is exact on $\mathcal{F}(\nabla)$

$$
\begin{array}{ccc}
\nabla(w) & \mapsto & \Delta_R(w) \\
T(w) & \mapsto & P_R(w)
\end{array}
$$

Connection to Ringel dual of $A := A_0(\mathfrak{g})$

- Important modules in $\mathcal{O}_0(\mathfrak{g})$

$$
\begin{array}{ccccc}
 & \nearrow & \{\text{ Verma module } \Delta(w)\ \} & \longleftrightarrow & \{\text{coVerma module } \nabla(w)\} \\
w \in W & \longleftrightarrow & \{\text{ simple module } S(w)\ \} & & \\
 & \searrow & \{\text{projective indec. } P(w)\ \} & \longleftrightarrow & \{\text{injective indec. } I(w)\} \\
 & \searrow & \{\text{tilting indec. } T(w)\ \} & &
\end{array}
$$

- $R(A) := \mathrm{End}_{\mathfrak{g}}\left(\bigoplus_{w\in W} T(w)\right)^{op}$ is Ringel dual

$$\boxed{(A, \leqslant) \cong (R(A), \geqslant) \text{ is quasi-hereditary}}$$

- $\mathrm{Hom}_{\mathfrak{g}}\left(\bigoplus_{w\in W} T(w), -\right) : \mathrm{mod} A \to \mathrm{mod} R(A)$ is exact on $\mathcal{F}(\nabla)$

$$
\begin{array}{ccc}
\nabla(w) & \mapsto & \Delta_R(w) \\
T(w) & \mapsto & P_R(w)
\end{array}
$$

Connection to Ringel dual of $A := A_0(\mathfrak{g})$

- Important modules in $\mathcal{O}_0(\mathfrak{g})$

$$\begin{tikzcd}[column sep=small]
 & \{\text{ Verma module } \Delta(w)\ \} \arrow[r, leftrightarrow] & \{\text{coVerma module } \nabla(w)\} \\
w \in W \arrow[ur, leftrightarrow] \arrow[r, leftrightarrow] \arrow[dr, leftrightarrow] \arrow[ddr, leftrightarrow] & \{\text{ simple module } S(w)\ \} & \\
 & \{\text{projective indec. } P(w)\ \} \arrow[r, leftrightarrow] & \{\text{injective indec. } I(w)\} \\
 & \{\text{tilting indec. } T(w)\ \} &
\end{tikzcd}$$

- $R(A) := \mathrm{End}_{\mathfrak{g}}\left(\bigoplus_{w\in W} T(w)\right)^{op}$ is Ringel dual

$$\boxed{(A, \leqslant) \cong (R(A), \geqslant) \text{ is quasi-hereditary}}$$

- $\mathrm{Hom}_{\mathfrak{g}}\left(\bigoplus_{w\in W} T(w), -\right) : \mathrm{mod} A \to \mathrm{mod} R(A)$ is exact on $\mathcal{F}(\nabla)$

$$\begin{array}{ccc} \nabla(w) & \mapsto & \Delta_R(w) \\ T(w) & \mapsto & P_R(w) \end{array}$$

- There exists a map $W \to W$ with $w \mapsto w$

Connection to Ringel dual of $A := A_0(\mathfrak{g})$

- Important modules in $\mathcal{O}_0(\mathfrak{g})$

$$\begin{tikzcd}
 & \{\text{ Verma module } \Delta(w)\ \} \arrow[r, leftrightarrow] \arrow[dl, leftrightarrow] & \{\text{coVerma module } \nabla(w)\} \\
w \in W \arrow[r, leftrightarrow] & \{\text{ simple module } S(w)\ \} & \\
 & \{\text{projective indec. } P(w)\ \} \arrow[r, leftrightarrow] \arrow[ul, leftrightarrow] & \{\text{injective indec. } I(w)\} \\
 & \{\text{tilting indec. } T(w)\ \} \arrow[uul, leftrightarrow] &
\end{tikzcd}$$

- $R(A) := \mathrm{End}_{\mathfrak{g}}\left(\bigoplus_{w\in W} T(w)\right)^{op}$ is Ringel dual

$$\boxed{(A, \leqslant) \cong (R(A), \geqslant) \text{ is quasi-hereditary}}$$

- $\mathrm{Hom}_{\mathfrak{g}}\left(\bigoplus_{w\in W} T(w), -\right) : \mathrm{mod}A \to \mathrm{mod}R(A)$ is exact on $\mathcal{F}(\nabla)$

$$\begin{array}{ccccc}
\nabla(w) & \mapsto & \Delta_R(w) & \leftrightsquigarrow & \Delta(w) \\
T(w) & \mapsto & P_R(w) & \leftrightsquigarrow & P(w)
\end{array}$$

- There exists a map $W \to W$ with $w \mapsto w$

Connection to Ringel dual of $A := A_0(\mathfrak{g})$

Connection to Ringel dual of $A := A_0(\mathfrak{g})$

$$\begin{array}{ccc} W & \longrightarrow & W \\ w & \longmapsto & \color{blue}{w} \end{array} \quad \text{bijective}$$

Connection to Ringel dual of $A := A_0(\mathfrak{g})$

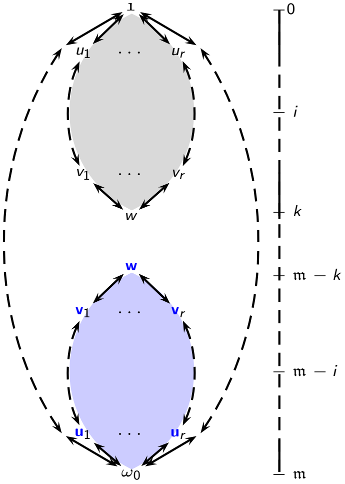
$$\begin{array}{ccc} W & \to & W \\ w & \mapsto & \color{blue}{w} \end{array} \text{ bijective with } \; w \leqslant v \; \text{ iff } \color{blue}{w} \geqslant \color{blue}{v}$$

Connection to Ringel dual of $A := A_0(\mathfrak{g})$

$$\begin{array}{ccc} W & \longrightarrow & W \\ w & \longmapsto & \color{blue}{w} \end{array} \text{ bijective with } \; w \leqslant v \; \text{ iff } \color{blue}{w} \geqslant \color{blue}{v} \quad \leftrightsquigarrow \quad \Lambda^{(w)} \mapsto \Lambda_{(\color{blue}{w})}$$

Connection to Ringel dual of $A := A_0(\mathfrak{g})$

$$\begin{array}{ccc} W & \to & W \\ w & \mapsto & w \end{array} \quad \text{bijective with} \quad w \leqslant v \ \text{ iff } \ w \geqslant v \quad \leftrightsquigarrow \quad \Lambda^{(w)} \mapsto \Lambda_{(w)}$$



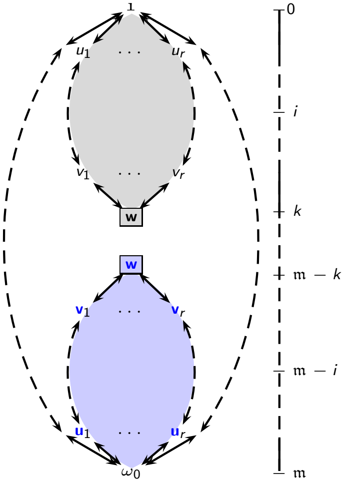
Connection to Ringel dual of $A := A_0(\mathfrak{g})$

$$\begin{array}{ccc} W & \to & W \\ {\color{blue}w} & \mapsto & w \end{array} \quad \text{bijective with} \quad w \leqslant v \quad \text{iff} \quad {\color{blue}w} \geqslant {\color{blue}v} \quad \leftrightsquigarrow \quad \Lambda^{(w)} \mapsto \Lambda_{({\color{blue}w})}$$

$$\left|\left\{ v \in \Lambda^{(w)} \; ; \; l(v) = i \right\}\right|$$

$$||$$

$$\left|\left\{ {\color{blue}v} \in \Lambda_{({\color{blue}w})} \; ; \; l({\color{blue}v}) = \mathfrak{m} - i \right\}\right|$$



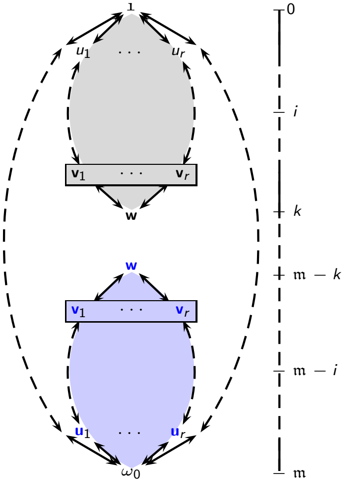
Connection to Ringel dual of $A := A_0(\mathfrak{g})$

$\begin{array}{ccc} W & \to & W \\ w & \mapsto & w \end{array}$ bijective with $w \leqslant v$ iff $w \geqslant v$ $\quad\leftrightsquigarrow\quad \Lambda^{(w)} \mapsto \Lambda_{(w)}$

$$\left|\left\{ v \in \Lambda^{(w)} \; ; \; l(v) = i \right\}\right|$$

$$||$$

$$\left|\left\{ v \in \Lambda_{(w)} \; ; \; l(v) = \mathfrak{m} - i \right\}\right|$$



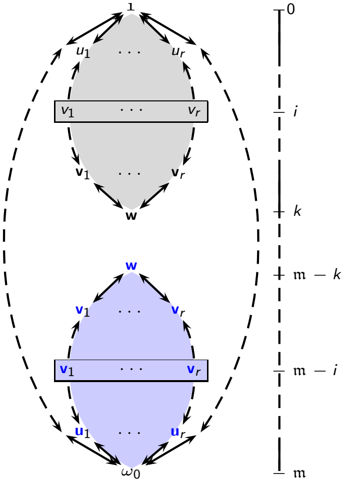
Connection to Ringel dual of $A := A_0(\mathfrak{g})$

$$\begin{array}{ccc} W & \rightarrow & W \\ w & \mapsto & w \end{array} \quad \text{bijective with} \quad w \leqslant v \quad \text{iff} \quad w \geqslant v \quad \leftrightsquigarrow \quad \Lambda^{(w)} \mapsto \Lambda_{(w)}$$

$$\left| \left\{ v \in \Lambda^{(w)} \; ; \; l(v) = i \right\} \right|$$

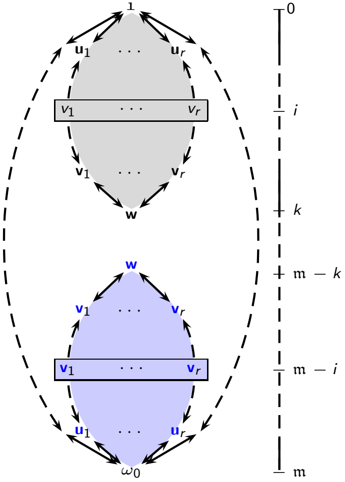
$$||$$

$$\left| \left\{ v \in \Lambda_{(w)} \; ; \; l(v) = \mathfrak{m} - i \right\} \right|$$



Connection to Ringel dual of $A := A_0(\mathfrak{g})$

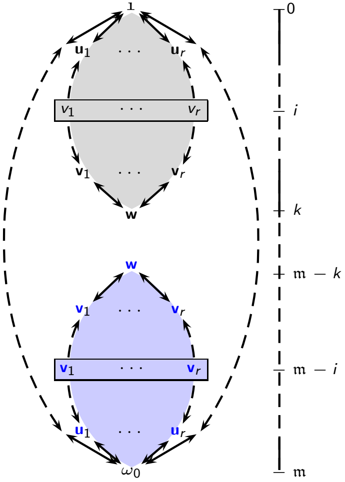
$$\begin{array}{ccc} W & \to & W \\ w & \mapsto & w \end{array} \quad \text{bijective with} \quad w \leqslant v \ \text{ iff } \ w \geqslant v \quad \leftrightsquigarrow \quad \Lambda^{(w)} \mapsto \Lambda_{(w)}$$



Connection to Ringel dual of $A := A_0(\mathfrak{g})$

$\begin{array}{ccc} W & \to & W \\ \mathbf{w} & \mapsto & w \end{array}$ bijective with $w \leqslant v$ iff $\mathbf{w} \geqslant \mathbf{v}$ $\leftrightsquigarrow$ $\Lambda^{(w)} \mapsto \Lambda_{(\mathbf{w})}$

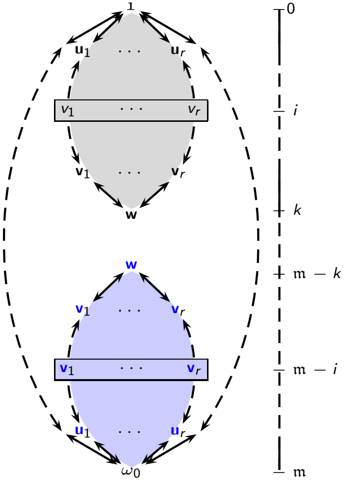
- $(T(\mathbf{w}) : \nabla(\mathbf{v})) = (P(w) : \Delta(v))$



Connection to Ringel dual of $A := A_0(\mathfrak{g})$

$$\begin{array}{ccc} W & \to & W \\ \color{blue}{w} & \mapsto & w \end{array} \quad \text{bijective with} \quad w \leqslant v \ \text{ iff } \ \color{blue}{w} \geqslant \color{blue}{v} \quad \leftrightsquigarrow \quad \Lambda^{(w)} \mapsto \Lambda_{(\color{blue}{w})}$$

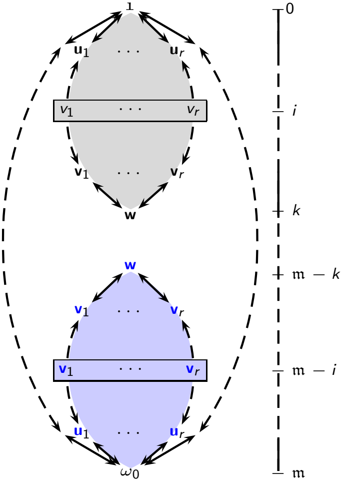
- $(T(\color{blue}{w}) : \nabla(\color{blue}{v})) = (P(w) : \Delta(v))$
- $\operatorname{soc} T(\color{blue}{w}) \cong \operatorname{soc} P(w)$



Connection to Ringel dual of $A := A_0(\mathfrak{g})$

$$\begin{array}{ccc} W & \to & W \\ \color{blue}{w} & \mapsto & w \end{array}$$ bijective with $w \leqslant v$ iff $\color{blue}{w} \geqslant \color{blue}{v}$ $\quad\leftrightsquigarrow\quad \Lambda^{(w)} \mapsto \Lambda_{(\color{blue}{w})}$

- $(T(\color{blue}{w}) : \nabla(\color{blue}{v})) = (P(w) : \Delta(v))$
- $\operatorname{soc} T(\color{blue}{w}) \cong \operatorname{soc} P(w)$

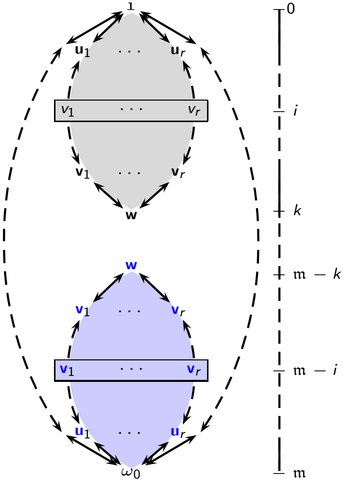


Lemma

Connection to Ringel dual of $A := A_0(\mathfrak{g})$

$$\begin{array}{ccc} W & \to & W \\ w & \mapsto & w \end{array} \quad \text{bijective with} \quad w \leqslant v \ \text{ iff } \ w \geqslant v \quad \leftrightsquigarrow \quad \Lambda^{(w)} \mapsto \Lambda_{(w)}$$

- $(T(w) : \nabla(v)) = (P(w) : \Delta(v))$
- $\operatorname{soc} T(w) \cong \operatorname{soc} P(w)$



Lemma

Let $w \in W$. The following statements are equivalent

Connection to Ringel dual of $A := A_0(\mathfrak{g})$

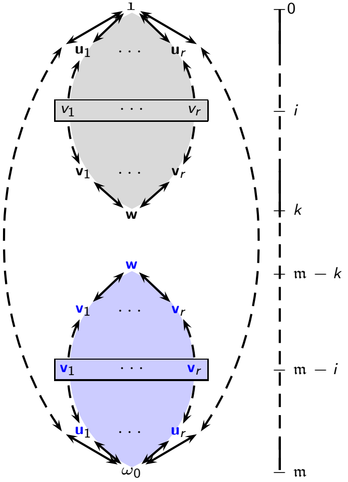
$\begin{array}{ccc} W & \to & W \\ w & \mapsto & w \end{array}$ bijective with $w \leqslant v$ iff $w \geqslant v$ $\rightsquigarrow$ $\Lambda^{(w)} \mapsto \Lambda_{(w)}$

- $(T(w) : \nabla(v)) = (P(w) : \Delta(v))$
- $\mathrm{soc}\, T(w) \cong \mathrm{soc}\, P(w)$

Lemma

Let $w \in W$. The following statements are equivalent

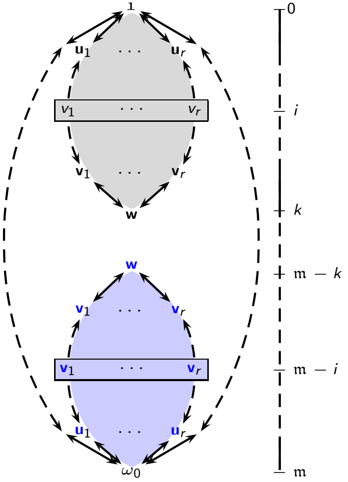
- $\mathrm{soc}\, T(w)$ *is simple*



Connection to Ringel dual of $A := A_0(\mathfrak{g})$

$$\begin{array}{ccc} W & \to & W \\ w & \mapsto & w \end{array} \quad \text{bijective with} \quad w \leqslant v \ \text{ iff } \ w \geqslant v \quad \leadsto \quad \Lambda^{(w)} \mapsto \Lambda_{(w)}$$

- $(T(w) : \nabla(v)) = (P(w) : \Delta(v))$
- $\operatorname{soc} T(w) \cong \operatorname{soc} P(w)$



Lemma

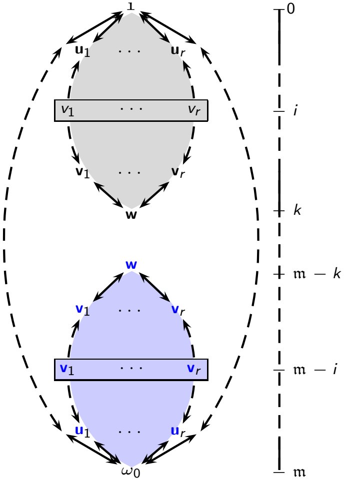
Let $w \in W$. The following statements are equivalent

- $\operatorname{soc} T(w)$ *is simple*
- $T(w) \cong P_{A(w)}(\omega_0)$

Connection to Ringel dual of $A := A_0(\mathfrak{g})$

$\begin{array}{ccc} W & \to & W \\ {\color{blue}w} & \mapsto & w \end{array}$ bijective with $w \leqslant v$ iff ${\color{blue}w} \geqslant {\color{blue}v}$ $\rightsquigarrow$ $\Lambda^{(w)} \mapsto \Lambda_{({\color{blue}w})}$

- $(T({\color{blue}w}) : \nabla({\color{blue}v})) = (P(w) : \Delta(v))$
- $\operatorname{soc} T({\color{blue}w}) \cong \operatorname{soc} P(w)$



Lemma

Let $w \in W$. The following statements are equivalent

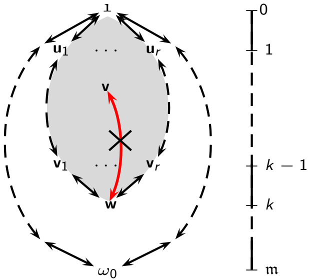
- $\operatorname{soc} T({\color{blue}w})$ *is simple*
- $T({\color{blue}w}) \cong P_{A({\color{blue}w})}(\omega_0)$
- $T({\color{blue}w}) \cong I_{A({\color{blue}w})}(\omega_0)$

Main Theorem

Theorem

Let $w \in W$. The following statements are equivalent

- $\mu(w, v) = 0$ *for all* $v \in \Lambda^{(w)}$ *with* $w \ntriangleleft v$;
- $\left|\{v \in \Lambda^{(w)}\ ;\ l(v) = i\}\right| = \left|\{v \in \Lambda^{(w)}\ ;\ l(v) = l(w) - i\}\right| \ \forall i$
- $(P(w) : \Delta(v)) = 1$ *for all* $v \in \Lambda^{(w)}$
- $\operatorname{soc} P(w)$ *is simple*



Main Theorem

Theorem

Let $w \in W$. The following statements are equivalent

- $\mu(w, v) = 0$ *for all* $v \in \Lambda^{(w)}$ *with* $w \ntriangleleft v$;
- $\left|\left\{v \in \Lambda^{(w)} \; ; \; l(v) = i\right\}\right| = \left|\left\{v \in \Lambda^{(w)} \; ; \; l(v) = l(w) - i\right\}\right| \; \forall i$
- $(P(w) : \Delta(v)) = 1$ *for all* $v \in \Lambda^{(w)}$
- $\operatorname{soc} P(w)$ *is simple*

Main Theorem

Theorem

Let $w \in W$. The following statements are equivalent

- $\mu(w, v) = 0$ *for all* $v \in \Lambda^{(w)}$ *with* $w \ntriangleleft v$;
- $\left|\{v \in \Lambda^{(w)}\ ;\ l(v) = i\}\right| = \left|\{v \in \Lambda^{(w)}\ ;\ l(v) = l(w) - i\}\right|\ \forall i$
- $(P(w) : \Delta(v)) = 1$ *for all* $v \in \Lambda^{(w)}$
- $\operatorname{soc} P(w)$ *is simple*
- $\operatorname{soc} T(w)$ *is simple*

Main Theorem

Theorem

Let $w \in W$. The following statements are equivalent

- $\mu(w,v) = 0$ *for all* $v \in \Lambda^{(w)}$ *with* $w \not\lhd v$;
- $\left|\{v \in \Lambda^{(w)} \ ; \ l(v) = i\}\right| = \left|\{v \in \Lambda^{(w)} \ ; \ l(v) = l(w) - i\}\right| \ \forall i$
- $(P(w) : \Delta(v)) = 1$ *for all* $v \in \Lambda^{(w)}$
- $\operatorname{soc} P(w)$ *is simple*
- $\operatorname{soc} T(w)$ *is simple*
- $(T(w) : \nabla(v)) = 1$ *for all* $v \in \Lambda_{(w)}$

Main Theorem

Theorem

Let $w \in W$. The following statements are equivalent

- $\mu(w, v) = 0$ *for all* $v \in \Lambda^{(w)}$ *with* $w \ntriangleleft v$;
- $\left|\{v \in \Lambda^{(w)} \ ; \ l(v) = i\}\right| = \left|\{v \in \Lambda^{(w)} \ ; \ l(v) = l(w) - i\}\right| \ \forall i$
- $(P(w) : \Delta(v)) = 1$ *for all* $v \in \Lambda^{(w)}$
- $\operatorname{soc} P(w)$ *is simple*
- $\operatorname{soc} T(w)$ *is simple*
- $(T(w) : \nabla(v)) = 1$ *for all* $v \in \Lambda_{(w)}$
- $T(w) \cong P_{A(w)}(\omega_0)$

Main Theorem

Theorem

Let $w \in W$. The following statements are equivalent

- $\mu(w, v) = 0$ *for all* $v \in \Lambda^{(w)}$ *with* $w \ntriangleleft v$;
- $\left|\{v \in \Lambda^{(w)}\ ;\ l(v) = i\}\right| = \left|\{v \in \Lambda^{(w)}\ ;\ l(v) = l(w) - i\}\right|\ \forall i$
- $(P(w) : \Delta(v)) = 1$ *for all* $v \in \Lambda^{(w)}$
- $\mathrm{soc}\, P(w)$ *is simple*
- $\mathrm{soc}\, T(w)$ *is simple*
- $(T(w) : \nabla(v)) = 1$ *for all* $v \in \Lambda_{(w)}$
- $T(w) \cong P_{A(w)}(\omega_0)$
- $T(w) \cong I_{A(w}(\omega_0)$

Main Theorem

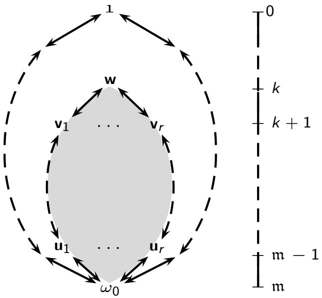
Main Theorem

Theorem

Main Theorem

Theorem

Let $w \in W$. The following statements are equivalent

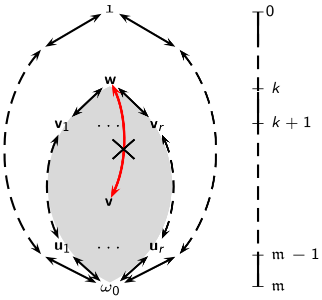


Main Theorem

Theorem

Let $w \in W$. The following statements are equivalent

- $\mu(w, v) = 0$ *for all* $v \in \Lambda_{(w)}$ *with* $w \not\triangleright v$;

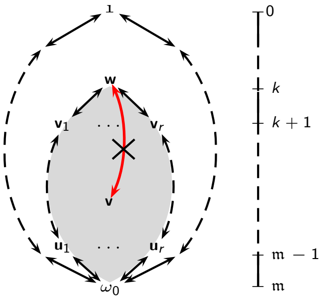


Main Theorem

Theorem

Let $w \in W$. The following statements are equivalent

- $\mu(w, v) = 0$ *for all* $v \in \Lambda_{(w)}$ *with* $w \not\triangleright v$;
- $\left|\{v \in \Lambda_{(w)} \mid l(v) = l(w) + i\}\right| = \left|\{v \in \Lambda_{(w)} \mid l(v) = \mathfrak{m} - i\}\right|$



Main Theorem

Proposition

Main Theorem

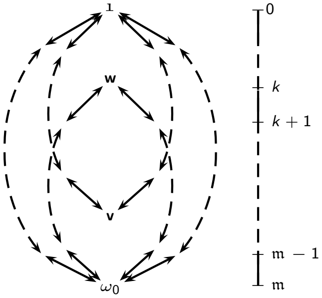
Proposition

Let $w \in W$ *and* $v \in \Lambda_{(w)}$ *with* $w \not\triangleright v$.

Main Theorem

Proposition

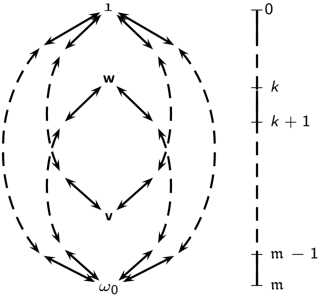
Let $w \in W$ and $v \in \Lambda_{(w)}$ with $w \not\triangleright v$.



Main Theorem

Proposition

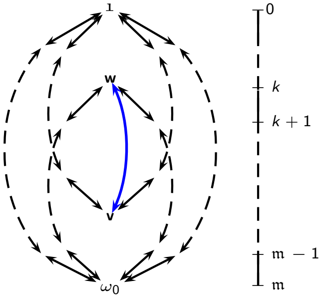
Let $w \in W$ *and* $v \in \Lambda_{(w)}$ *with* $w \not\triangleright v$. *If* $\mu(w, v) \neq 0$



Main Theorem

Proposition

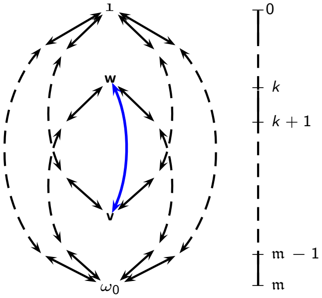
Let $w \in W$ *and* $v \in \Lambda_{(w)}$ *with* $w \not\triangleright v$. *If* $\mu(w, v) \neq 0$



Main Theorem

Proposition

Let $w \in W$ and $v \in \Lambda_{(w)}$ with $w \not\triangleright v$. If $\mu(w,v) \neq 0$ then there exist $i, j \in \mathbb{N}$ with $0 \leq i \leq l(v)$ and $l(w) \leq j \leq \mathfrak{m}$ such that

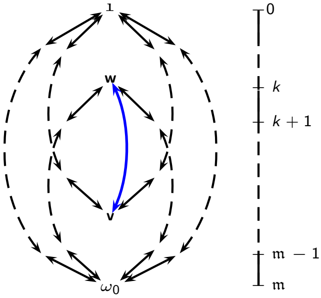


Main Theorem

Proposition

Let $w \in W$ and $v \in \Lambda_{(w)}$ with $w \not\triangleright v$. If $\mu(w,v) \neq 0$ then there exist $i, j \in \mathbb{N}$ with $0 \leq i \leq l(v)$ and $l(w) \leq j \leq \mathfrak{m}$ such that

1. $\left|\{u \in \Lambda^{(v)} \; ; \; l(u) = l(v) - i\}\right| \neq \left|\{u \in \Lambda^{(v)} \; ; \; l(u) = i\}\right|$



Main Theorem

Proposition

Let $w \in W$ and $v \in \Lambda_{(w)}$ with $w \not\triangleright v$. If $\mu(w,v) \neq 0$ then there exist $i, j \in \mathbb{N}$ with $0 \leq i \leq l(v)$ and $l(w) \leq j \leq \mathfrak{m}$ such that

1. $\left|\left\{u \in \Lambda^{(v)} \; ; \; l(u) = l(v) - i\right\}\right| \neq \left|\left\{u \in \Lambda^{(v)} \; ; \; l(u) = i\right\}\right|$
2. $\left|\left\{u \in \Lambda_{(w)} \mid l(u) = l(w) + j\right\}\right| \neq \left|\left\{u \in \Lambda_{(w)} \mid l(u) = \mathfrak{m} - j\right\}\right|$