Universal Deformation Rings and Fusion

David Meyer

University of Iowa

Maurice Auslander Distinguished Lectures and International Conference

Universal Deformation Rings

Let Γ be a finite group, V an absolutely irreducible $\mathbb{F}_p\Gamma$-module. By Mazur's work, V has a well-defined universal deformation ring $R(\Gamma, V)$ which is universal with respect to all lifts of V over complete local commutative Noetherian rings with residue field $\mathbb{F}_p$.

Theorem

By Mazur, if $\dim_{\mathbb{F}_p}(\mathrm{H}^1(\Gamma, \mathrm{Hom}_{\mathbb{F}_p}(V, V))) = r$ and $\dim_{\mathbb{F}_p}(\mathrm{H}^2(\Gamma, \mathrm{Hom}_{\mathbb{F}_p}(V, V))) = s$, then:

- $R(\Gamma, V) \cong \mathbb{Z}_p[\![t_1, t_2, ... t_r]\!]/I$ where r is minimal and
- I is an ideal whose minimal numbers of generators is bounded above by s.

General Setting

- p a prime
- G a finite p'-group
- Γ an extension of G by $N := \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$
- Assume that $\mathbb{F}_p$ is a splitting field of G.

We have a short exact sequence

$$0 \to \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z} \to \Gamma \to G \cong \Gamma/N \to 1$$

1. $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$ is a 2-dimensional $\mathbb{F}_p$ representation of G denoted by ϕ.
2. Let V be a 2-dimensional irreducible $\mathbb{F}_p G$-module inflated to Γ.

Question

What is the relationship between the fusion of $N = \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$ in Γ and $\mathrm{H}^2(\Gamma, \mathrm{Hom}_{\mathbb{F}_p}(V, V))$, resp. $\mathrm{R}(\Gamma, V)$?

Cohomology

Let $\tilde{\phi}$ denote the contragredient of ϕ and let $W_{\tilde{\phi}}$ (resp. $W_{\det\circ(\tilde{\phi})}$) denote the $\mathbb{F}_p\Gamma$-module associated to $\tilde{\phi}$ (resp. $\det\circ(\tilde{\phi})$).

Theorem

Using the above notation,
$\mathrm{H}^2(\Gamma, \mathrm{Hom}_{\mathbb{F}_p}(V, V)) \cong [(W_{\tilde{\phi}} \otimes V^* \otimes V) \oplus (W_{\det\circ(\tilde{\phi})} \otimes V^* \otimes V)]^{\Gamma/N}$.

This result provides a way of using character theory to compute the first and second cohomology group of Γ with coefficients in $\mathrm{Hom}_{\mathbb{F}_p}(V, V)$. To prove the theorem we need the following result.

Lemma

Using the above notation, for all $i \geq 1$,
$H^i(N, V^* \otimes V) \cong V^* \otimes V \otimes H^i(N, \mathbb{F}_p)$ as $\mathbb{F}_p\Gamma/N$-modules, and
$\mathrm{H}^i(\Gamma, V^* \otimes V) \cong \mathrm{H}^0(\Gamma/N, \mathrm{H}^i(N, V^* \otimes V)) \cong [\mathrm{H}^i(N, V^* \otimes V)]^G$.

Cohomology

Proof of the Theorem. By the lemma, $\mathrm{H}^2(N, \mathrm{Hom}_{\mathbb{F}_p}(V, V)) \cong \mathrm{Hom}_{\mathbb{F}_p}(V, V) \otimes \mathrm{H}^2(N, \mathbb{Z}/p\mathbb{Z})$ as $\mathbb{F}_p\Gamma/N$-modules.

Consider the Kummer sequence $1 \to \mu_p \xrightarrow{\iota} \mathbb{C}^* \xrightarrow{p} \mathbb{C}^* \to 1$, where $\mathbb{C}^* \xrightarrow{p} \mathbb{C}^*$ denotes the map given by $z \xrightarrow{p} z^p$. We consider this sequence as a sequence of $\mathbb{Z}N$-modules with trivial N-action.

Applying the functor $\mathrm{Hom}_{\mathbb{Z}\,\Gamma/N}(\mathbb{Z}, -)$ we obtain the long exact sequence

$$\ldots \xrightarrow{\delta} \mathrm{H}^1(N, \mu_p) \xrightarrow{\iota_*} \mathrm{H}^1(N, \mathbb{C}^*) \xrightarrow{p_*} \mathrm{H}^1(N, \mathbb{C}^*) \xrightarrow{\delta} \mathrm{H}^2(N, \mu_p) \xrightarrow{\iota_*}$$

$$\mathrm{H}^2(N, \mathbb{C}^*) \xrightarrow{p_*} \mathrm{H}^2(N, \mathbb{C}^*) \xrightarrow{\delta} \mathrm{H}^3(N, \mu_p) \xrightarrow{\iota_*} \ldots$$

Since N is elementary abelian, $\mathrm{H}^i(N, \mathbb{C}^*) \xrightarrow{p_*} \mathrm{H}^i(N, \mathbb{C}^*)$ is trivial. Thus, we get the short exact sequence of $\mathbb{F}_p\Gamma/N$-modules

$$0 \to \mathrm{H}^1(N, \mathbb{C}^*) \xrightarrow{\delta} \mathrm{H}^2(N, \mathbb{Z}/p\mathbb{Z}) \xrightarrow{\iota_*} \mathrm{H}^2(N, \mathbb{C}^*) \to 0.$$

Applying the functor $\mathrm{Hom}_{\mathbb{F}_p}(V, V) \otimes -$, and taking fixed points, we obtain

$\mathrm{H}^2(\Gamma, \mathrm{Hom}_{\mathbb{F}_p}(V, V)) \cong$
$[\mathrm{H}^1(N, \mathbb{C}^*) \otimes \mathrm{Hom}_{\mathbb{F}_p}(V, V)]^{\Gamma/N} \oplus [\mathrm{H}^2(N, \mathbb{C}^*) \otimes \mathrm{Hom}_{\mathbb{F}_p}(V, V)]^{\Gamma/N}$.

Therefore, our result follows once we show that $\mathrm{H}^1(N, \mathbb{C}^*) \cong W_{\tilde{\phi}}$ and $\mathrm{H}^2(N, \mathbb{C}^*) \cong W_{\det \circ \tilde{\phi}}$ as $\mathbb{F}_p\Gamma/N$- modules.

Since N is an elementary abelian p-group which acts trivially on $\mathbb{C}^*$, $\mathrm{H}^1(N, \mathbb{C}^*) = \mathrm{Hom}(N, \mathbb{C}^*) \cong \mathrm{Hom}_{\mathbb{F}_p}(N, \mathbb{F}_p)$ as $\mathbb{F}_p G$-modules, which implies $\mathrm{H}^1(N, \mathbb{C}^*) \cong W_{\tilde{\phi}}$. It remains to determine the Γ/N-module structure of $\mathrm{H}^2(N, \mathbb{C}^*)$. Our result follows after a quick computation, using that $\mathrm{H}^2(N, \mathbb{C}^*) = N \wedge N$. $\square$

Cohomology

So we have shown

Theorem

$\mathrm{H}^2(\Gamma, \mathrm{Hom}_{\mathbb{F}p}(V, V)) \cong [(W_{\tilde{\phi}} \otimes V^* \otimes V) \oplus (W_{\mathrm{det} \circ (\tilde{\phi})} \otimes V^* \otimes V)]^{\Gamma/N}$.

Additionally, we have

Corollary

Under the same hypotheses

(a) $\mathrm{H}^1(\Gamma, \mathrm{Hom}_{\mathbb{F}_p}(V, V)) = (W_{\tilde{\phi}} \otimes V^* \otimes V)^{\Gamma/N}$

(b) $\mathrm{H}^1(\Gamma, \mathrm{Hom}_{\mathbb{F}_p}(V, V))$ is a summand of $\mathrm{H}^2(\Gamma, \mathrm{Hom}_{\mathbb{F}_p}(V, V))$

(c) $\dim_{\mathbb{F}_p}(\mathrm{H}^1(\Gamma, \mathrm{Hom}_{\mathbb{F}_p}(V, V))) \leq \dim_{\mathbb{F}_p}(\mathrm{H}^2(\Gamma, \mathrm{Hom}_{\mathbb{F}_p}(V, V)))$

Definitions

Let N, Γ, G, ϕ be as above.

1. For every irreducible $\mathbb{F}_p G$-module V, let $d_V^i = \dim_{\mathbb{F}_p}(\mathrm{H}^i(\Gamma, \mathrm{Hom}_{\mathbb{F}_p}(V, V))$ for i=1,2. Note that this number depends on ϕ.

2. We say an irreducible $\mathbb{F}_p G$-module V_0 is **cohomologically maximal** for ϕ if $d_{V_0}^2$ is maximal among all d_V^2. Similarly, we say an irreducible representation ρ of G over $\mathbb{F}_p$ is **cohomologically maximal** for ϕ if ρ corresponds to a $\mathbb{F}_p G$-module with this property.

3. We call the orbits of the action ϕ of G on N the **fusion orbits** of ϕ. For all $m \geq 1$, let $F_{\phi,m}$ be the number of fusion orbits of ϕ with cardinality m. Then, the sequence $\{F_{\phi,m}\}_{m\geq 1}$ is called the **fusion** of ϕ.

Main Result

We now consider the case where G is dihedral of order $2n$.

Let $\mathrm{Rep}_2(G)$ be a complete set of representatives of isomorphism classes of all 2-dimensional representations of G over $\mathbb{F}_p$ and let $\mathrm{Irr}_2(G) \subset \mathrm{Rep}_2(G)$ be the subset of isomorphism classes of irreducible 2-dimensional representations.

For ρ in $\mathrm{Irr}_2(G)$, let V_ρ be an irreducible $\mathbb{F}_p G$-module with representation ρ.

Main Theorem

Assuming the above notation, there exists a subset Ω of $\mathrm{Irr}_2(G)$, and a map of sets $\mathrm{T} : \mathrm{Irr}_2(G) \to \mathrm{Rep}_2(G)$ such that the following is true.

(a) If n is odd, then $\Omega = \mathrm{Irr}_2(G)$ and T is a bijection. If n is even, $\Omega = \mathrm{Irr}_2(G) \cap \mathrm{T}(\mathrm{Irr}_2(G))$ and for all ψ in Ω, $|\,\mathrm{T}^{-1}(\psi)\,| = 2$.

(b) If $\phi \in \Omega$, then the fusion of ϕ is uniquely determined by the set $\{ker(\rho) \mid \rho \in \mathrm{Irr}_2(G)$ is cohomologically maximal for $\phi\} = \{ker(\rho) \mid \rho \in \mathrm{Irr}_2(G)$ with $R(\Gamma, V_\rho) \not\cong \mathbb{Z}_p\}$.

Main Result

So for ϕ in Ω, the fusion can be detected by the set

$\{ker(\rho) \mid \rho \in \mathrm{Irr}_2(G)$ is cohomologically maximal for $\phi\} =$

$\{ker(\rho) \mid \rho \in \mathrm{Irr}_2(G)$ with $R(\Gamma, V_\rho) \ncong \mathbb{Z}_p\}$.

Moreover, we have the following.

Proposition

Let $G = D_{2n}$.

1. If n is odd, $\phi_1, \phi_2 \in \Omega$. Then ϕ_1 and ϕ_2 have the same fusion if and only if $\mathrm{T}^{-1}(\phi_1)$ and $\mathrm{T}^{-1}(\phi_2)$ have the same kernel.
2. If n is even, $\phi_1, \phi_2 \in \Omega$. Then ϕ_1 and ϕ_2 have the same fusion if and only if $\{$kernel of ψ: $\psi \in \mathrm{T}^{-1}(\phi_1)\}$=$\{$kernel of ψ: $\psi \in \mathrm{T}^{-1}(\phi_2)\}$.
3. If ϕ is in Ω, then $V = V_\psi$ is cohomologically maximal for ϕ if and only if $\mathrm{T}(\psi) = \phi$.

Cohomology for $G = D_{2n}$

For $G = D_{2n}$, all 2-dim. irreducible representations are of the form:

$r \xrightarrow{\theta_i} \begin{pmatrix} \omega^i & 0 \\ 0 & \omega^{-i} \end{pmatrix}$, $s \xrightarrow{\theta_i} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, For $i = 1 < \frac{n}{2}$, ω a primitve nth root of unity in $\mathbb{F}_p^*$, $p \equiv 1 \text{ mod}(n)$.

Note: $\theta_i = \mathrm{Ind}_{\langle r \rangle}^{G}(\chi_i)$, where χ_i is the one-dimensional representation of $\langle r \rangle$ with $\chi_i(r) = \omega^i$.

We define the map $\mathrm{T} : \mathrm{Irr}_2(G)$ to $\mathrm{Rep}_2(G)$ by $\mathrm{T}(\theta_i) = \mathrm{T}(\mathrm{Ind}_{\langle r \rangle}^{G}(\chi_i)) = \mathrm{Ind}_{\langle r \rangle}^{G}(\chi_{2i})$.

Cohomology for $G = D_{2n}$

Proposition

Let $G = D_{2n}$. Let Ω be as before.

1. If n is odd, then $\mathrm{T} : \mathrm{Irr}_2(G) \to \mathrm{Irr}_2(G) = \Omega, \mathrm{T}$ a bijection, and for any ϕ irreducible, there exists a unique $\psi = \mathrm{T}^{-1}(\phi)$ irreducible with $d^2_{V_\psi} = 2$. For all other V, $d^2_V = 1$. So V_ψ is cohomologically maximal for ϕ.
2. For n even, then $\mathrm{T} : \mathrm{Irr}_2(G) \to \mathrm{Rep}_1(G)$, and for any ϕ in Ω, there exist exactly two V_ψ irreducible with $d^2_{V_\psi} = 2$. For all other V, $d^2_V = 1$. Thus, there are precisely two ψ that are cohomologically maximal for ϕ. These exceptional representations are exactly the elements of $\mathrm{T}^{-1}(\{\phi\})$.

Universal Deformation Rings

Using a result of Bleher, Chinburg, de Smit we can show the following.

Proposition

Let $G = D_{2n}$. Let $\mathrm{R} = \mathrm{R}(\Gamma, V)$, let ϕ be in Ω. Then, for V in $\mathrm{Irr}_2(G)$, the universal deformation ring

$$\mathrm{R} = \begin{cases} \mathbb{Z}_p & \text{if V is not cohomologically maximal for } \phi \\ \mathbb{Z}_p[[t]]/(t^2, tp) & \text{if V is cohomologically maximal for} \phi \end{cases}$$

Additionally, $\mathrm{R}(\Gamma, V) = \mathbb{Z}_p[[t]]/(t^2, tp)$ if and only if d_V^2 is equal to two.

V is cohomologically maximal for ϕ if and only if $d_V^1 = 1$ and $d_V^2 = 2$. Otherwise, $d_V^1 = 0$ and $d_V^2 = 1$.

Fusion for Dihedral Groups

We compute the fusion for $N = \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$ in Γ.

$\Gamma/N = D_{2n}$, $p \equiv 1 \pmod{n}$, $\phi = \theta_i$.

Fusion

Elements of N are fused if and only if they are in the same ϕ orbit.

The cardinality of the orbits are as follows:

$$|Orbit((x,y))| = \begin{cases} 1, & \text{if } (x,y) = 0 \\ n/\gcd(i,n), & \text{if } (x,y) \in {\mathbb{F}_p}^* \times {\mathbb{F}_p}^*, y/x \in \langle \omega^i \rangle \\ 2n/\gcd(i,n), & \text{otherwise} \end{cases}$$

Thus, fusion is determined by $\gcd(i,n)$.

Thank You!