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Derived representation type of Schur superalgebras

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Representation type of an associative algebra

A finite-dimensional associative algebra A over an algebraically closed field K is one of the following types:
- finite representation type
- tame representation type
- wild representation type

Modules over Schur algebra $S(n, d)$ correspond to polynomial representations of the general linear group $GL(n)$ of degree d.

Representation type of $S(n, d)$ was determined by Doty, Erdmann, Martin and Nakano (1999). There are many cases of all representation types (finite, tame and wild).

Derived representation type of an associative algebra

Instead of asking how many non-isomorphic indecomposable objects are there in the category of modules over A, look at indecomposable objects (up to translations) in the bounded derived category of A.

Bekkert and Drozd (2004) proved that every algebra A is one of the following types:

- derived finite representation type
- derved tame representation type
- derived wild representation type

Derived representation type of Schur algebra $S(n,d)$

Let p be the characteristic of the field K. Bekkert-Futorny (2003) proved that $S(n,d)$ is derived tame (**finite**) if and only if one of the following holds:

(a) p=0 or p $>$ d).

(b) p=2, n=2, d=3).

c) $p = 2, n = 2, d = 5$ or $d = 7$.

d) $n = 2, p \leq d < 2p$ and $d \neq 3$ if $p = 2$.

e) $p = 2, n \geq 3, d = 2$ or $d = 3$.

f) $p = 3, n = 3, d = 4$ or $d = 5$.

Schur superalgebra

K - algebraically closed field of characteristic $p \neq 2$.

Fix an alphabet consisting of m even symbols $1, \ldots, m$ and n odd symbols $m+1, \ldots, m+n$ and declare the parity $|c_{ij}|$ of c_{ij} as the sum of parities of i and j. Elements $c_{ij}, 1 \leq i, j \leq m+n$, are subject to supercommutativity relation

$$c_{ij} c_{kl} = (-1)^{|c_{ij}||c_{kl}|} c_{kl} c_{ij}.$$

Denote by $A(m|n)$ a commutative superalgebra freely generated by elements $c_{ij}, 1 \leq i,j \leq m+n$. $A(m|n)$ is a superbialgebra with respect to the comultiplication $\Delta(c_{ij}) = \sum_{1\leq k\leq m+n} c_{ik} \otimes c_{kj}$ and counit $\epsilon(c_{ij}) = \delta_{ij}$.

The algebra $A(m|n) = \oplus_{r\geq 0} A(m|n,r)$ has obvious grading where every homogeneous component $A(m|n,r)$ is a supercoalgebra.

Its dual $S(m|n,r) = A(m|n,r)^*$ is a Schur superalgebra.

General linear supergroup

Write a generic matrix $C = (c_{ij})_{1\leq i,j\leq m+n}$ as a block matrix

$$\begin{pmatrix} C_{00} & C_{01} \\ C_{10} & C_{11} \end{pmatrix},$$

where entries of C_{00} and C_{11} are even and entries of C_{01} and C_{10} are odd. The localization of $A(m|n)$ by elements $det(C_{00})$ and $det(C_{11})$ is the coordinate superalgebra $K[G]$ of the general linear supergroup $G = GL(m|n)$. The superalgebra G has a diagonally embedded even subalgebra $G_{ev} \cong GL(m) \times GL(n)$.

The category of modules over $S(m|n, r)$ is isomorphic to the category of polynomial representations of degree r over the general linear supergroup $GL(m|n)$.

Representation type of Schur superalgebra

Hemmer, Kujawa and Nakano (2006) proved that $S(m|n,d)$ has finite representation type (**is semisimple**) if and only if one of the following holds:

a) p=0 or d<p

b) m=n=1 and p does not divide d

c) $p \leq d < 2p$

d) $m = n = 1$ and p divides d.

In all other cases $S(m|n,r)$ has wild representation type.

Derived representation type of Schur superalgebra

Theorem: (Futorny - M., 2012)

a) Assume that $S(m|n,d)$ is semisimple, that is, one of the following conditions is satisfied:

(i) $p = 0$

(ii) $d < p$

(iii) $m = n = 1$ and p does not divide d.

Then $S(m|n,d)$ is of derived finite representation type.

b) If $S(m|n,d)$ is not semisimple, then $S(m|n,d)$ is of derived wild representation type.

Define a class of quiver algebras F_k by the quiver

$$\begin{array}{ccccccccc} 1 & \overset{\alpha_1}{\longrightarrow} & 2 & \overset{\alpha_2}{\longrightarrow} & 3 & & k-1 & \overset{\alpha_{k-1}}{\longrightarrow} & k \\ \bullet & \overset{\beta_1}{\longleftarrow} & \bullet & \overset{\beta_1}{\longleftarrow} & \bullet & \ldots & \bullet & \overset{\beta_{k-1}}{\longleftarrow} & \bullet \end{array}$$

and relations

$$\alpha_i\alpha_{i+1} = 0, \beta_{i+1}\beta_i = 0, \alpha_1\beta_1 = 0, \beta_{k-1}\alpha_{k-1} = 0, \beta_i\alpha_i = \alpha_{i+1}\beta_{i+1},$$

where $i = 1, \ldots, k-2$.

If $k > 3$, then there is a complex $N^\bullet$ of $F_k - k\langle x, y\rangle$-bimodules such that the functor $N^\bullet \otimes_{k\langle x,y\rangle} -$ from $k\langle x, y\rangle$-modules to projective F_k- complexes preserves indecomposability and isomorphism classes. Therefore algebra F_k for $k > 3$ is derived wild representation type.

We denote by $\mathbf{P}_i$ the indecomposable projective corresponding to the vertex $i \in Q_0$ and by $p(w)$ the morphism between two indecomposable projectives corresponding to the path w of Q.

$d_i = \dim M(i)$ and denote by $[M(x)]$ the matrix corresponding to the map $M(x) : M(s(x)) \to M(e(x))$ with respect to some fixed basis.

For $P^\bullet \in \mathbf{C}^{-,b}(A-\text{pro})\backslash \mathbf{C}^b(A-\text{pro})$, let s be the maximal number such that $P^s \neq 0$ and $H^i(P^\bullet)=0$ for $i \leq s$. Then, $\alpha(P^\bullet)^\bullet$ denotes the *brutal truncation* of $P^\bullet$ below s (see Weibel, 1994), i.e., the complex given by

$$\alpha(P^\bullet)^i = \begin{cases} P^i, & \text{if } i \geq s; \\ 0, & \text{otherwise,} \end{cases}$$

$$\partial^i_{\alpha(P^\bullet)^\bullet} = \begin{cases} \partial^i_{P^\bullet}, & \text{if } i \geq s; \\ 0, & \text{otherwise.} \end{cases}$$

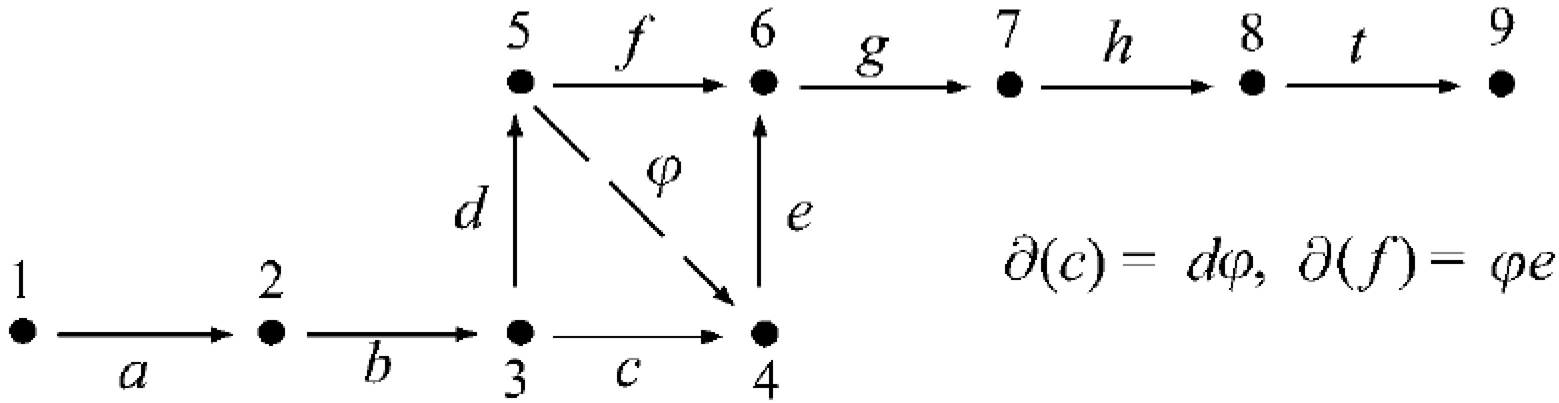
For $P^\bullet \neq 0^\bullet \in \mathbf{C}^b(A-\mathrm{pro})$, let t be the maximal number such that $P^i = 0$ for $i < t$. Then, $\beta(P^\bullet)^\bullet$ denotes the (*good*) *truncation* of $P^\bullet$ below t (see Weibel, 1994), i.e., the complex given by

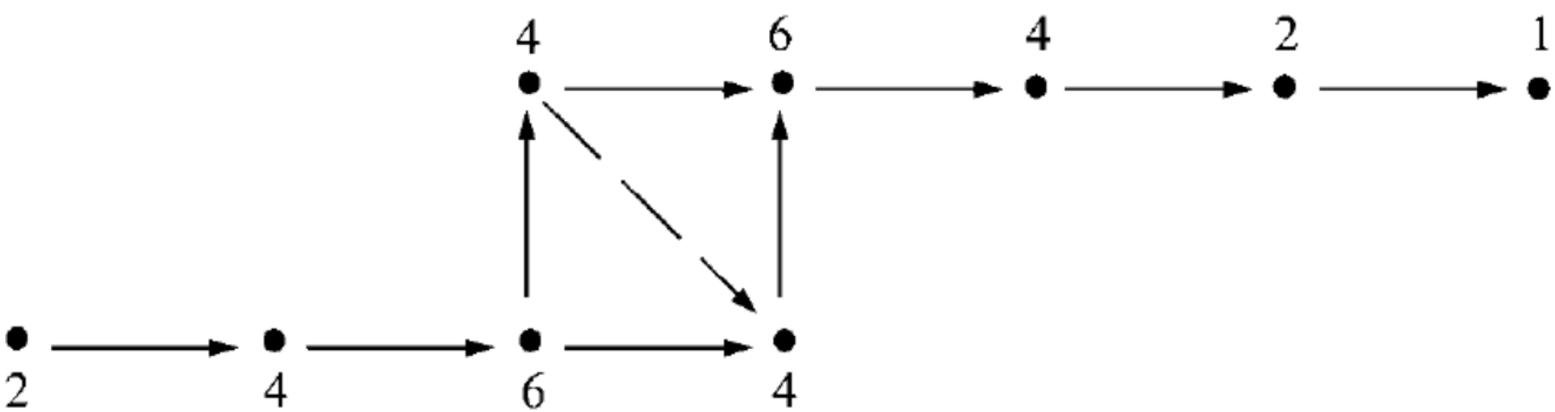
$$\beta(P^\bullet)^i = \begin{cases} P^i, & \text{if } i \geq t; \\ \operatorname{Ker} \partial^t_{P^\bullet}, & \text{if } i = t-1; \\ 0, & \text{otherwise,} \end{cases}$$

$$\partial^i_{\beta(P^\bullet)^\bullet} = \begin{cases} \partial^i_{P^\bullet}, & \text{if } i \geq t; \\ i_{\operatorname{Ker} \partial^t_{P^\bullet}}, & \text{if } i = t-1; \\ 0, & \text{otherwise,} \end{cases}$$

where $i_{\operatorname{Ker} \partial^t_{P^\bullet}}$ is the obvious inclusion.

W :





Since box W is wild, there exists $W-k\langle x, y\rangle$-bimodule M such that the functor $M \otimes_{k\langle x,\ y\rangle} -$ preserves indecomposability and isomorphism classes. Denote by $N^{\bullet}$ the following complex of $A-k\langle x, y\rangle$-bimodules.

Set $N^i = \mathbf{P}_3^{d_i}$ for $1 \leq i \leq 3$, $N^j = \mathbf{P}_2^{d_{j+1}}$ for $5 \leq j \leq 8$, $N^4 = \mathbf{P}_2^{d_5} \oplus \mathbf{P}_3^{d_4}$,

$$\partial^1 = p(\beta_2\alpha_2)[M(a)], \partial^2 = p(\beta_2\alpha_2)[M(b)],$$

$$\partial^3 = (p(\beta_2)[M(d)]p(\beta_2\alpha_2)[M(c)]),$$

$$\partial^4 = (p(\alpha_2\beta_2)[M(f)]p(\beta_2)[M(e)])^T, \partial^5 = p(\alpha_2\beta_2)[M(g)],$$

$$\partial^6 = p(\alpha_2\beta_2)[M(h)], \partial^7 = p(\alpha_2\beta_2)[M(t)].$$

Thank you!