Repns of Modular Skew group algebras

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I: Notation:

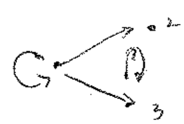
- $k = \bar{k}$, $\mathrm{Char}\, k = p \geq 0$
- A: a $< \infty$ diml k-algebra
- G: a finite gp, each $g \in G$ acts on A as algebra automorphism
- Skew gp algebra $AG = A \otimes_k kG$ as spaces, multiplication

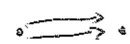
defined by $ag \cdot bh = a\, g(b)\, gh$.

Example: 1) A  $G = C_p = \langle x \rangle$ acts on it by $x(n \bmod p) = (n+1) \bmod p$

Then AG is $M_{n\times n}(k)$.

2)  $C_2 = \langle g \rangle$, g permutes 2 and 3

 which is Morita equivalent to 

3)  $C_2 = \langle g \rangle \Rightarrow$ 

II: Backgroup:

If $|G|$ is invertible. Then:

(Reiten, Riedtmann): 1) $\mathrm{gldim}\, AG < \infty \Leftrightarrow \mathrm{gldim}\, A < \infty$

2) AG has $< \infty$ repn type $\Leftrightarrow$ So does A.

3) AG is self-injective $\Leftrightarrow$ So is A

(Martinez): If G preserves grading. AG is Kosml $\Leftrightarrow$ A is Kosml.

Moreover, the Yoneda algebra of AG is also a skew gp algebra

2

5. (20 points) Use the Intermediate Value Theorem to show that the equation $e^x = 3x$ has a solution in $[0, 1]$ and a solution in $[1, 2]$. $(2.7 < e < 2.8)$.

(Dionne, Lanzilotta, Smith): A is piecewise hereditary $\Rightarrow AG$ is piece h.

Question: For general gps G, under what conditions ~~these results~~ do A and AG still share these properties?

III: Induction and Restriction.

Let $H \leq G$, then ΛH is a subalgebra of ΛG.

Define:

$\uparrow_H^G : \Lambda H\text{-mod} \longrightarrow \Lambda G\text{-mod}, \quad N \longmapsto \Lambda G \otimes_{\Lambda H} N;$

$\downarrow_H^G : \Lambda G\text{-mod} \longrightarrow \Lambda H\text{-mod} \quad M \longmapsto {}_{\Lambda H}M$

They perserves proj modules, exact.

Prop: Let $H \leq G$.

1) If ΛG has ~~[illegible]~~ one of the following properties, so does ΛH.

i) ~~[illegible]~~ gldim $< \infty$

ii) fi. dim $< \infty$

iii) finite repn type

iv) self-injective.

v) s. gldim $< \infty$

2) Moreover, if $|G:H|$ is invertible in k, then

i) $\operatorname{gldim} AG = \operatorname{gldim} AH$

ii) $\operatorname{fidim} AG = \operatorname{fidim} AH$

iii) $\operatorname{s.gl.dim} AG = \operatorname{s.gldim} AH$

(is self-inj)

iv) AG has $<\infty$ repn type iff

so does AH. (so is AH)

IV: Action of p-gps.

So we know AG and AH share many properties where $H \leq G$ is a Sylow p-subgp.

Let $S \leq G$ be a p-group and consider ΛH. We assume:

(*) A has a set of primitive orthogonal idempotents $E = \{e_i\}_{i \in [n]}$ s.t. E is closed under the action of S.

Prop: Let $S \leq G$, as above

1) If $\operatorname{gldim} AS < \infty \Rightarrow$ the action of S on E is free.

2) Suppose that $p \neq 2, 3$, A is not local. If AS has $<\infty$ repn type $\Rightarrow$ the action of S on E is free.

Prop: Suppose the action of S on E is free. Then

1) AS is Morita equivalent to A^S,

2) $AS \cong A^{|S|}$ and A is a proj ΛS-module

3) An AS-module M is proj (inj) $\Leftrightarrow$ ${}_A M$ is proj (inj).

V: Main Result

<u>Thm</u>: A, G as before. Suppose that A has a set of primitive orthogonal idempotents $E = \{e_i\}_{i \in [n]}$ which is closed under the action of a Sylow p-subgp $S \leq G$. Then:

1) $\operatorname{gldim} AG < \infty \Leftrightarrow \operatorname{gldim} A < \infty$, and the action of S on E is free. If $\operatorname{gldim} AG < \infty$, then $\operatorname{gldim} AG = \operatorname{gldim} A$.

2) $p \neq 2, 3$, A is not local. AG has $< \infty$ repn type $\Leftrightarrow$ so does A, and the action of S on E is free.

3) If $\operatorname{fin\,dim} AG < \infty \Rightarrow \operatorname{fin\,dim} A < \infty$. If the action of S on E is free, then $\operatorname{fin\,dim} AG = \operatorname{fin\,dim} A$.

4) For $s.\operatorname{gldim}$, we have the same result described in 1).

Corollary: A. G. S. E as above. AG is piecewise hereditary iff so is A and the action of S on E is free.

<u>Thm (Happel/Zacharia)</u>: A is piecewise hereditary $\Leftrightarrow$ $s.\operatorname{gldim} A < \infty$.

VI: Application to transporter category

$\mathcal{P}$: a finite poset.

G: acts on $\mathcal{P}$.

The Grothendieck construction $\mathcal{T} = G \propto \mathcal{P}$ is called a transporter category.

Its category algebra $k\mathcal{T}$ is a skew gp algebra.

Why is it called transporter?

Thm: Suppose that $p \neq 2, 3$. Then $\mathcal{T} = G \propto \mathcal{P}$ is of finite repn type if and only if one of the following is true:

1) $|G|$ is invertible, and $\mathcal{P}$ is of $< \infty$ repn type;

2) $\mathcal{P}$ has only one element and G is of $< \infty$ repn type.

VII: Koszul properties

$A = \bigoplus_{i \geq 0} A_i$ locally finite, generated in degrees 0 and 1.

G: preserves grading.

Then AG is locally finite, generated in degrees 0 and 1 with $(AG)_0 = A_0 G$, which in general is not semisimple.

Last time I have introduced a generalised Koszul theory which do not ~~required~~ require the degree 0 component to be semisimple.

Using that, we prove

Thm: Let M be a graded AG-module, $\Gamma = \mathrm{Ext}^*_A(M, M)$.

1) $M \otimes_k kG \in AG$-mod is generalized Koszl $\Leftrightarrow$ ${}_A M$ is generalized Koszl. In particular, AG is a generalized Koszl algebra iff so is A.

2) If M is generalized Koszl, then $\mathrm{Ext}^*_{AG}(M \otimes kG, M \otimes kG)$ is ~~skew gp algebra~~ isomorphic to the skew gp algebra ΓG.

VII: Question

We know regular gp algebra kG. findim $kG = 0$.

Question: If findim $A < \infty$, then findim $AG < \infty$?