Invariant Theory of AS-Regular Algebras: A Survey

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Invariants under S_n
Permutations of $x_1, \cdots, x_n$.

(Painter: Christian Albrecht Jensen) (Wikepedia)

Gauss' Theorem

The subring of invariants under S_n is a polynomial ring

$$k[x_1, \cdots, x_n]^{S_n} = k[\sigma_1, \cdots, \sigma_n]$$

where σ_ℓ are the n elementary symmetric functions for $\ell = 1, \ldots, n$, or the n power sums:

$$P_\ell = x_1^\ell + \cdots + x_i^\ell + \cdots + x_n^\ell$$

Question: When is $k[x_1, \cdots, x_n]^G$ a polynomial ring? (G a finite group of graded automorphisms.)

Shephard-Todd-Chevalley Theorem

Let k be a field of characteristic zero.

Theorem (1954). The ring of invariants $k[x_1, \cdots, x_n]^G$ under a finite group G is a polynomial ring if and only if G is generated by reflections.

A linear map g on V is called a reflection of V if all but one of the eigenvalues of g are 1, i.e. $\dim V^g = \dim V - 1$.

Example: Transposition permutation matrices are reflections, and S_n is generated by reflections.

When is $k[x_1, x_2, \ldots, x_n]^G$:

- A polynomial ring? Shephard-Todd-Chevalley Theorem (1954)

- A Gorenstein ring? Watanabe's Theorem (1974), Stanley's Theorem (1978) ($H_{A^G}(t^{-1}) = \pm t^m H_{A^G}(t)$).

  Example. Let $g = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ act on $k[x, y]$

  $$k[x, y]^g = k\langle x^2, xy, y^2 \rangle \cong \frac{k[a, b, c]}{\langle b^2 - ac \rangle}, \quad H(t) = \frac{1 + t^2}{(1 - t^2)^2}$$

- A complete intersection? Nakajima (1984), Gordeev (1986)

Noncommutative Generalizations

Replace $k[x_1, \cdots, x_n]$ by a connected graded noetherian Artin-Schelter regular algebra A. Let $k = \mathbb{C}$.

G a group of graded automorphisms of A.
Not all linear maps act on A.

Question: Under what conditions on G is A^G Artin-Schelter regular, or AS-Gorenstein, or a "complete intersection"?

More generally, consider finite dimensional (semisimple) Hopf algebras H acting on A.

Artin-Schelter Gorenstein/Regular

Noetherian connected graded algebra A is Artin-Schelter Gorenstein if:

- A has graded injective dimension $n < \infty$ on the left and on the right,
- $\text{Ext}^i_A(k, A) = \text{Ext}^i_{A^{op}}(k, A) = 0$ for all $i \neq n$, and
- $\text{Ext}^n_A(k, A) \cong \text{Ext}^n_{A^{op}}(k, A) \cong k(\ell)$ for some ℓ.

If in addition,

- A has finite (graded) global dimension, and
- A has finite Gelfand-Kirillov dimension,

then A is called Artin-Schelter regular of dimension n.

An Artin-Schelter regular graded domain A is called a quantum polynomial ring of dimension n if $H_A(t) = (1 - t)^{-n}$.

Graded automorphisms of $\mathbb{C}_q[x, y]$

If $q \neq \pm 1$ there are only diagonal automorphisms:

$$g = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}.$$

When $q = \pm 1$ there also are automorphisms of the form:

$$g = \begin{bmatrix} 0 & a \\ b & 0 \end{bmatrix}:$$

$$yx = qxy$$

$$g(yx) = g(qxy)$$

$$axby = qbyax$$

$$abxy = q^2 abxy$$

$$q^2 = 1.$$

Noncommutative Gauss' Theorem?

Example: $S_2 = \langle g \rangle$, for $g = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, acts on $A = \mathbb{C}_{-1}[x, y]$ and A^{S_2} is generated by

$$P_1 = x + y \text{ and } P_2 = x^3 + y^3$$

($x^2 + y^2 = (x + y)^2$ and $g \cdot xy = yx = -xy$ so no generators in degree 2). The generators are NOT algebraically independent. A^{S_2} is NOT AS-regular (but it is a hyperplane in an AS-regular algebra).

The transposition $(1, 2)$ is NOT a "reflection".

Definition of “reflection”: Want A^G AS-regular

All but one eigenvalue of g is 1 $\rightsquigarrow$

The trace function of g acting on A of dimension n has a pole of order $n-1$ at $t=1$, where

$$Tr_A(g,t) = \sum_{k=0}^{\infty} trace(g|A_k)t^k = \frac{1}{(1-t)^{n-1}q(t)} \text{ for } q(1) \neq 0.$$

Conjecture: A^G is AS-regular if and only if G is generated by “reflections”.

Examples $G =< g >$ on $A = \mathbb{C}_{-1}[x, y]$ $(yx = -xy)$:

(a) $g = \begin{bmatrix} \epsilon_n & 0 \\ 0 & 1 \end{bmatrix}$, $Tr(g, t) = \dfrac{1}{(1-t)(1-\epsilon_n t)}$, A^G AS-regular.

(b) $g = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $Tr(g, t) = \dfrac{1}{1+t^2}$, A^G not AS-regular.

(c) $g = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, $Tr(g, t) = \dfrac{1}{(1-t)(1+t)}$, A^G AS-regular.

$$A^G = \mathbb{C}[xy, x^2 + y^2].$$

For $A = \mathbb{C}_{q_{ij}}[x_1, \cdots, x_n]$ the groups generated by "reflections" are exactly the groups whose fixed rings are AS-regular rings.

What are the reflection groups?

For quantum polynomial rings they must be generated by classical reflections and "mystic" reflections.

Example: The "binary dihedral groups" of order 4ℓ generated by

$$g_1 = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \text{ and } g_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

for λ a primitive 2ℓth root of unity, acts on $A = \mathbb{C}_{-1}[x, y]$.

$$A^G = \mathbb{C}[xy,\ x^{2\ell} + y^{2\ell}].$$

Molien's Theorem: Using trace functions

$$\text{Jørgensen-Zhang:} \quad H_{A^G}(t) = \frac{1}{|G|} \sum_{g \in G} Tr_A(g, t)$$

.

Example (c) $A = \mathbb{C}_{-1}[x, y]$ and $g = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

$\sigma_1 = x^2 + y^2$, $\sigma_2 = xy$ and $A^G \cong \mathbb{C}[\sigma_1, \sigma_2]$.

$$H_{A^G}(t) = \frac{1}{4(1-t)^2} + \frac{2}{4(1-t^2)} + \frac{1}{4(1+t)^2} = \frac{1}{(1-t^2)^2}.$$

Algebra of Covariants

Theorem (Chevalley-Serre). If G acts on $A = \mathbb{C}[x_1, \ldots, x_n]$ with θ_i a set of n homogeneous algebraically independent G-invariants of $\mathbb{C}[x_1, \ldots, x_n]$, and if $I = \langle \theta_1, \ldots, \theta_n \rangle$, then A/I, as a G-module, is isomorphic to t copies of the regular representation of G, where

$$t = \prod_i \frac{\deg(\theta_i)}{|G|}$$

(when G is generated by reflections then $t = 1$).

Theorem. Let A be AS-regular of GKdim A = n with Hilbert series $1/((1-t)^n p(t))$. If there are n homogeneous G-invariant elements θ_i with θ_i normal in A and θ_i regular on $A/\langle\theta_1, \ldots, \theta_{i-1}\rangle$, then for $I = \langle\theta_1, \ldots, \theta_n\rangle$ as a G-module, A/I is a isomorphic to t copies of the regular representation, where

$$t = \prod_i \frac{\deg(\theta_i)}{|G|(p(1))}$$

(when G is generated by reflections then $t = 1$).

Example 1. Binary dihedral groups on $A = \mathbb{C}_{-1}[x, y]$ with

$$A^G = \mathbb{C}[xy,\ x^{2\ell} + y^{2\ell}].$$

$\mathbb{C}_{-1}[x, y]/(xy,\ x^{2\ell} + y^{2\ell})$ is one copy of regular representation of G.

Example 2. S_n acting on $A = \mathbb{C}_{-1}[x_1, \ldots, x_n]$ with θ_i the ith symmetric function in the $\{x_i^2\}$ – e.g. n=2
$\mathbb{C}_{-1}[x, y]/\langle x^2 + y^2, x^2y^2\rangle$ is $(2 \cdot 4)/2 = 4$ copies of the regular representation of S_2.

Invariants under Hopf Algebra Actions

Let (H, Δ, ϵ, S) be a Hopf algebra and A be a Hopf-module algebra so

$$h \cdot (ab) = \sum (h_1 \cdot a)(h_2 \cdot b) \quad \text{and} \quad h \cdot 1_A = \epsilon(h) 1_A$$

for all $h \in H$, and all $a, b \in A$.
The invariants of H on A are

$$A^H := \{a \in A \mid h \cdot a = \epsilon(h)a \text{ for all } h \in H\}.$$

When $H = k[G]$ and $\Delta(g) = g \otimes g$ then $g \cdot (ab) = g(a)g(b)$.

Etingof and Walton (2013): Let H be a finite dimensional semisimple Hopf algebra over a field of characteristic zero, and let A be a commutative domain. If A is an H-module algebra for an inner faithful action of H on A, then H is a group algebra.

Question: Under what conditions on H is A^H an AS-regular algebra?

When is H a “quantum reflection group”?

Kac/Masuoka's 8-dimensional semisimple Hopf algebra

H_8 is generated by x, y, z with the following relations:

$$x^2 = y^2 = 1, \;\; xy = yx, \;\; zx = yz,$$

$$zy = xz, \;\; z^2 = \frac{1}{2}(1 + x + y - xy).$$

$$\Delta(x) = x \otimes x, \quad \Delta(y) = y \otimes y,$$

$$\Delta(z) = \frac{1}{2}(1 \otimes 1 + 1 \otimes x + y \otimes 1 - y \otimes x)(z \otimes z),$$

$$\epsilon(x) = \epsilon(y) = \epsilon(z) = 1, \quad S(x) = x^{-1}, \; S(y) = y^{-1}, \; S(z) = z.$$

H_8 has a unique irreducible 2-dimensional representation on $\mathbb{C}u + \mathbb{C}v$ given by

$$x \mapsto \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad y \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad z \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

Example 1: Let $A = \mathbb{C}\langle u, v\rangle / \langle u^2 - v^2 \rangle$.
$A^H = \mathbb{C}[u^2, (uv)^2 - (vu)^2]$, a commutative polynomial ring.
H is "quantum reflection group" for A.

Example 2: Let $A = \mathbb{C}\langle u, v\rangle / \langle vu - iuv \rangle$. $A^H = \mathbb{C}[u^2v^2, u^2 + v^2]$, a commutative polynomial ring.
H is "quantum reflection group" for A.

H not semisimple

The Sweedler algebra $H(-1)$ generated by g and x

$$g^2 = 1, \quad x^2 = 0, \quad xg = -gx$$

$$\Delta(g) = g \otimes g \quad \Delta(x) = g \otimes x + x \otimes 1,$$

$$\epsilon(g) = 1, \epsilon(x) = 0 \quad S(g) = g, \; S(x) = -gx.$$

Then $H(-1)$ acts on $k[u, v]$ as

$$x \mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad g \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$k[u, v]^{H(-1)} = k[u, v^2].$$

Questions:

When is A^H regular?

Are the trace functions useful in understanding when H is a "quantum reflection group"? What are the elements whose traces determine if H is a "quantum reflection group"?

Gorenstein Invariant Subrings
Watanabe's Theorem (1974):

If G is a finite subgroup of $SL_n(k)$ then $k[x_1, \cdots, x_n]^G$ is Gorenstein.

If A is AS-regular, when is A^G AS-Gorenstein?

What is the generalization of determinant = 1?

Trace Functions and Homological Determinant

When A is AS-regular of dimension n, then when the trace is written as a Laurent series in t^{-1}

$$Tr_A(g,t) = (-1)^n (\text{hdet } g)^{-1} t^{-\ell} + \text{higher terms}$$

(Jing-Zhang)

Generalized Watanabe's Theorem (Jørgensen-Zhang): A^G is AS-Gorenstein when all elements of G have homological determinant 1.

If g is a 2-cycle and $A = \mathbb{C}_{-1}[x_1 \ldots, x_n]$ then

$$Tr_A(g, t) = \frac{1}{(1+t^2)(1-t)^{n-2}}$$

$$= (-1)^n \frac{1}{t^n} + \text{ higher terms}$$

so hdet $g = 1$, and for ALL groups G of $n \times n$ permutation matrices, A^G is AS-Gorenstein. Not true for commutative polynomial ring – e.g.

$$\mathbb{C}[x_1, x_2, x_3, x_4]^{\langle(1,2,3,4)\rangle}$$

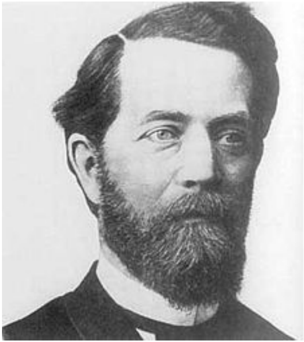
is not Gorenstein, while

$$\mathbb{C}_{-1}[x_1, x_2, x_3, x_4]^{\langle(1,2,3,4)\rangle}$$

is AS-Gorenstein.

Binary Polyhedral Groups

Felix Klein (1884)



Classified the finite subgroups of $SL_2(k)$, for k an algebraically closed field of char 0, and calculated invariants $k[u, v]^G$.

Actions of Binary Polyhedral Groups on $k[u, v]$

G a finite subgroup of $SL_2(k)$

$k[u, v]^G$ is a hypersurface ring
$k[u, v]^G \cong k[x, y, z]/(f(x, y, z))$,
a “Kleinian singularity”, of type A,D or E (corresponding to the type of McKay quiver of the irreducible representations of the group G).

The Homological Determinant of a Hopf Action

Since $\text{Ext}^n_A(k, A)$ is 1-dimensional, the left H-action on $\text{Ext}^n_A(k, A)$ defines an algebra map $\eta' : H \to k$ such that $h \cdot \mathbf{e} = \eta'(h)\mathbf{e}$ for all $h \in H$.

The homological determinant hdet is equal to $\eta' \circ S$, where S is the antipode of H.

The homological determinant is trivial if $\text{hdet} = \epsilon$.

Actions of Quantum Binary Polyhedral Groups on Quantum Planes

Find all H, a finite dimensional Hopf algebra acting on A, an AS-regular algebra of dimension 2:

$$k_J[u,v] := k\langle u,v\rangle/(vu - uv - u^2)$$

$$\text{or} \qquad k_q[u,v] := k\langle u,v\rangle/(vu - quv),$$

with trivial homological determinant, so that A is an H module algebra, the action is inner faithful and preserves the grading.

Use the classification of finite Hopf quotients of the coordinate Hopf algebra $\mathcal{O}_q(SL_2(k))$ (Bichon-Natale, Müller, Stefan).

AS reg alg A gldim 2	f.d. Hopf algebra(s) H acting on A
$k[u,v]$	$k\tilde{\Gamma}$
$k_{-1}[u,v]$	kC_n for $n \geq 2$; kD_{2n}; $(kD_{2n})^\circ$; $\mathcal{D}(\tilde{\Gamma})^\circ$ for $\tilde{\Gamma}$ nonabelian
$k_q[u,v]$, q root of 1, $q^2 \neq 1$ if U non-simple if U simple, $o(q)$ odd if U simple, $o(q)$ even, and $q^4 \neq 1$ if U simple, $q^4 = 1$	 kC_n for $n \geq 3$; $(T_{q,\alpha,n})^\circ$; $1 \to (k\tilde{\Gamma})^\circ \to H^\circ \to \mathfrak{u}_q(\mathfrak{sl}_2)^\circ \to 1$; $1 \to (k\Gamma)^\circ \to H^\circ \to \mathfrak{u}_{2,q}(\mathfrak{sl}_2)^\circ \to 1$; $1 \to (k\Gamma)^\circ \to H^\circ \to \mathfrak{u}_{2,q}(\mathfrak{sl}_2)^\circ \to 1$ $1 \to (k\Gamma)^\circ \to H^\circ \to \frac{\mathfrak{u}_{2,q}(\mathfrak{sl}_2)^\circ}{(e_{12}-e_{21}e_{11}^2)} \to 1$
$k_q[u,v]$, q not root 1	$kC_n, n \geq 2$
$k_J[u,v]$	kC_2

Commutative Complete Intersections

Theorem (Gulliksen) (1971):
Let A be a connected graded noetherian commutative algebra. Then the following are equivalent.

1. A is isomorphic to $k[x_1, x_2, \ldots, x_n]/(d_1, \ldots, d_m)$ for a homogeneous regular sequence.
2. The Ext-algebra $\mathrm{Ext}^*_A(k, k)$ is noetherian.
3. The Ext-algebra $\mathrm{Ext}^*_A(k, k)$ has finite GK-dimension.

Noncommutative Complete Intersections

Let A be a connected graded finitely generated algebra.

1. We say A is a *classical complete intersection* if there is a connected graded noetherian AS regular algebra R and a sequence of regular normal homogeneous elements $\{\Omega_1, \cdots, \Omega_n\}$ of positive degree such that A is isomorphic to $R/(\Omega_1, \cdots, \Omega_n)$.
2. We say A is a *complete intersection of noetherian type* if the Ext-algebra $\mathrm{Ext}^*_A(k, k)$ is noetherian.
3. We say A is a *complete intersection of growth type* if the Ext-algebra $\mathrm{Ext}^*_A(k, k)$ has finite Gelfand-Kirillov dimension.
4. We say A is a *weak complete intersection* if the Ext-algebra $\mathrm{Ext}^*_A(k, k)$ has subexponential growth.

Noncommutative case:

Classical C.I.		C.I. of Noetherian Type
$\Downarrow$		$\Downarrow$
C.I. of Growth Type	$\Longrightarrow$	*Weak C.I.*

Noncommutative case:

Classical C.I.	$\not\Longleftarrow$	C.I. of Noetherian Type
$\Downarrow \quad \not\Uparrow$	$\not\nearrow$	$\Downarrow$
C.I. of Growth Type	$\Longrightarrow$	*Weak C.I.*

A^G a complete intersection:

Theorem: (Kac and Watanabe – Gordeev) (1982). If $\mathbb{C}[x_1, \ldots, x_n]^G$ is a complete intersection then G is generated by bi-reflections (all but two eigenvalues are 1).

For an AS-regular algebra A a graded automorphism g is a "bi-reflection" of A if

$$Tr_A(g, t) = \sum_{k=0}^{\infty} trace(g|A_k)t^k$$

$$= \frac{1}{(1-t)^{n-2}q(t)},$$

n = GKdim A, and $q(1) \neq 0$.

Example: A^G a complete intersection

$A = \mathbb{C}_{-1}[x, y, z]$ is regular of dimension 3, and

$$g = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

acts on it. The eigenvalues of g are $-1, i, -i$ so g is not a bi-reflection of A_1. However,
$Tr_A(g, t) = 1/((1+t)^2(1-t)) = -1/t^3 +$ higher degree terms and g is a “bi-reflection” with hdet $g = 1$.

$$A^g \cong \frac{k[X, Y, Z, W]}{\langle W^2 - (X^2 + 4Y^2)Z \rangle},$$

a commutative complete intersection.

Invariants A^G

Classical C.I.		C.I. of Noetherian Type
$\Downarrow$		$\Downarrow$
C.I. of Growth Type	$\Longrightarrow$	*Weak C.I.*
		$\Downarrow$
		Cyclotomic Gorenstein $H_{A^G}(t) = p(t)/q(t)$

?? $\Longrightarrow$ generated by quasi-bireflections

Gauss' Theorem

Invariants of $\mathbb{C}_{-1}[x_1, \ldots, x_n]$ under the full Symmetric Group S_n:

$\mathbb{C}_{-1}[x_1, \ldots, x_n]^{S_n}$ and $\mathbb{C}_{-1}[x_1, \ldots, x_n]^{A_n}$ are classical complete intersections.

Permutations in S_n are "bi-reflections" if and only if they are 2-cycles or 3-cycles.

Theorem. Let $A = k_{-1}[x_1, \cdots, x_n]$ and G be a finite subgroup of permutations of $\{x_1, \cdots, x_n\}$. If G is generated by quasi-bireflections then A^G is a classical complete intersection.

Question: Is the converse true?

Graded Down-up Algebras $A(\alpha,\beta), \beta \neq 0$:

Theorem. Let A be a down-up algebra with $\beta \neq 0$

$$(y^2x = \alpha yxy + \beta xy^2 \text{ and } yx^2 = \alpha xyx + \beta x^2y)$$

and G be a finite subgroup of graded automorphisms of A. Then the following are equivalent:

- A^G is a growth type complete intersection.
- A^G is cyclotomic Gorenstein and G is generated by quasi-bireflections.
- A^G is cyclotomic Gorenstein.

Question: Are these A^G also classical complete intersections?

Veronese Subrings

For a graded algebra A the rth Veronese $A^{\langle r\rangle}$ is the subring generated by all monomials of degree r.

If A is AS-Gorenstein of dimension n, then $A^{\langle r\rangle}$ is AS-Gorenstein if and only if r divides ℓ where $\text{Ext}^n_A(k, A) = k(\ell)$ (Jørgensen-Zhang).

Let $g = \text{diag}(\lambda, \cdots, \lambda)$ for λ a primitive rth root of unity; $G = (g)$ acts on A with $A^{\langle r\rangle} = A^G$.

If the Hilbert series of A is $(1-t)^{-n}$ then

$$Tr_A(g^i, t) = \frac{1}{(1-\lambda^i t)^n}.$$

For $n \geq 3$ the group $G = (g)$ contains no "bi-reflections", so $A^G = A^{\langle r\rangle}$ should not be a complete intersection.

Theorem:

Let A be noetherian connected graded algebra.

Suppose the Hilbert series of A is $(1-t)^{-n}$. If $r \geq 3$ or $n \geq 3$, then $H_{A^{\langle r \rangle}}(t)$ is not cyclotomic. Consequently, $A^{\langle r \rangle}$ is not a complete intersection of any type.

Auslander's Theorem

Let G be a finite subgroup of $GL_n(k)$ that contains no reflections, and let $A = k[x_1, \ldots, x_n]$. Then the skew-group ring $A \# G$ is isomorphic to $\mathrm{End}_{A^G}(A)$ as rings.

Question: Does Auslander's Theorem generalize to our context?