On support varieties over complete intersections

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The General Idea of Support Varieties

$$M \rightsquigarrow \mathcal{V}(M)$$

- Associate to an R-module M and algebraic set in some affine (or projective) space whose properties reflect homological characteristics of M.

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- Associate to an R-module M and algebraic set in some affine (or projective) space whose properties reflect homological characteristics of M.
- Throughout, R ring, $k = \bar{k}$, M, N f.g. R-modules.

The Typical Situation

Let $A = \oplus_{i \geq 0} A^i$ be a commutative graded ring with $A^i = 0$ for i odd. Suppose for every M there is a homomorphism of graded algebras

$$\eta_M : A \longrightarrow \operatorname{Ext}^*_R(M, M)$$

such that for every N and $\xi \in \operatorname{Ext}^*_R(M, N)$ we have

$$\xi \cdot \eta_M(a) = \eta_N(a) \cdot \xi \quad \text{for every} \quad a \in A$$

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Then A is called a *ring of central cohomology operators*.

Support and Varieties

The *cohomological support* of (M, N) is

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When A finitely generated over A^2 with $A^0 = k$, then the *support variety* of (M, N) is

$$\mathcal{V}_A(M,N) = (\operatorname{Supp}_A(M,N) \cap \operatorname{MaxSpec} A) \cup \{A^{\geq 1}\}$$

and $\mathcal{V}_A(M) = \mathcal{V}_A(M,k)$.

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- Finite dimensional algebras; A is the even part of the Hochschild cohomology ring.
- Complete intersections; A is a subring of the cohomology ring, generated by central elements of degree 2.

Special case: complete Intersections

Now assume that Q is a local (meaning also Noetherian) ring with maximal ideal $\mathfrak{n}$ and residue field k, $R = Q/(\boldsymbol{f})$ where $\boldsymbol{f} = f_1, \ldots, f_c$ is a regular sequence in $\mathfrak{n}^2$.

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In this case we have

$$A = R[\chi_1, \ldots, \chi_c]$$

as the ring of cohomology operators, defined from the Eisenbud operators 1980. (deg $\chi_i = 2$, $1 \leq i \leq c$)

Example

For $Q = k[[x, y]]$, $R = Q/(x^2, y^2)$, and $M = k$, the Eisenbud operators are defined by ...

A theorem of Gulliksen 1974 tells us when $\operatorname{Ext}_R^*(M, N)$ is a finitely generated graded module over $R[\chi_1, \ldots, \chi_c]$

Theorem

If $\operatorname{Ext}_Q^*(M, N)$ is finitely generated over R, then $\operatorname{Ext}_R^*(M, N)$ is a finitely generated graded module over $R[\chi_1, \ldots, \chi_c]$.

Fact: the action of A on $\operatorname{Ext}^*_R(M,k)$ factors through the algebra $\bar{A} = A \otimes_R k = k[\chi_1, \ldots, \chi_c]$, so we have the support variety $V_{\bar{A}}(M)$. In other words

$$\mathcal{V}_{\bar{A}}(M) = \{(b_1, \ldots, b_c) \in k^c \mid \phi(b_1, \ldots, b_c) = 0 \text{ for all } \phi \in \operatorname{Ann}_{\bar{A}} \operatorname{Ext}^*_R(M,k)\}$$

a closed set (cone) in k^c when $\operatorname{Ext}^*_R(M,k)$ is f.g. — e.g. Q is a regular local ring.

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Recall: if $\mathcal{M}$ is finitely generated and graded over $k[\chi_1, \ldots, \chi_c]$, then $b_i = \dim_k \mathcal{M}_i$ grows polynomially.

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One has

- $\mathcal{V}_{\bar{A}}(M, N) = \mathcal{V}_{\bar{A}}(M) \cap \mathcal{V}_{\bar{A}}(N)$
- For $0 \to M_1 \to M_2 \to M_3 \to 0$ one has

$$\mathcal{V}_{\bar{A}}(M_r) \subseteq \mathcal{V}_{\bar{A}}(M_s) \cup \mathcal{V}_{\bar{A}}(M_t)$$

  for $\{r, s, t\} = \{1, 2, 3\}$.
- $\mathcal{V}_{\bar{A}}(M) = \mathcal{V}_{\bar{A}}(\Omega M)$

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  - Further realizability ... of modules! Avramov and Jorgensen 201n.
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- $\mathcal{V}_{\bar{A}}(M) \cap \mathcal{V}_{\bar{A}}(N) = \{0\} \Leftrightarrow \mathrm{Ext}_R^{\gg 0}(M, N) = 0 \Leftrightarrow \mathrm{Tor}^R_{\gg 0}(M, N) = 0 \Leftrightarrow \mathrm{Ext}_R^{\gg 0}(N, M) = 0$

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The solution is suggested by a theorem of Avramov and Buchweitz (2000).

Definition

Let $R = Q/(\boldsymbol{f})$ with Q regular, and $V = (\boldsymbol{f})/\,\mathfrak{n}(\boldsymbol{f})$. Then

$$\mathcal{V}_R(M) = \{f + \mathfrak{n}(\boldsymbol{f}) \in V \mid \operatorname{pd}_{Q/(f)} M = \infty\}$$

is the support variety of M .

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Theorem

- *$\mathcal{V}_R(M)$ is well-defined.*
- *$\mathcal{V}_R(M)$ is an algebraic set (cone) in V.*

$\mathcal{V}_{\bar{A}}(M)$ agrees with $\mathcal{V}_R(M)$ ... a theorem of Avramov and Buchweitz 2000.

Intermediate Complete Intersections

Let W be a subspace of $V = (\boldsymbol{f})/\mathfrak{n}(\boldsymbol{f})$. Choose a basis $g_1 + \mathfrak{n}(\boldsymbol{f}), \ldots, g_d + \mathfrak{n}(\boldsymbol{f})$ of W. Then $R_W = Q/(g_1, \ldots, g_d)$ is an intermediate complete intersection

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Proof.

$$f + \mathfrak{n}(\boldsymbol{g}) \in \mathcal{V}_{R_W}(M) \Leftrightarrow \operatorname{pd}_{Q/(f)} M = \infty \Leftrightarrow f + \mathfrak{n}(\boldsymbol{f}) \in \mathcal{V}_R(M) \cap W$$

□

Example

For $c = 3$, assume that $\mathcal{V}_R(M) = \mathrm{Z}(\chi_1^2 + \chi_2^2 - \chi_3^2)$.

- $W = \langle f_1, f_3 \rangle$ then $\mathcal{V}_{R_W}(M)$ two transverse lines.
- $W = \langle f_2 \rangle$ then $\mathcal{V}_{R_W}(M) = \{0\}$.
- $W = \langle f_1 + f_3 \rangle$ then $\mathcal{V}_{R_W}(M) = W$.

Theorem

Let $W \subseteq V$. Let M_1 be an R_W-module and M_2 be an $R_{W^\perp}$-module (both MCM). Then

$$\mathcal{V}_{R_W}(M_1) = \mathcal{V}_R(M_1 \otimes_Q M_2) \cap W$$

and

$$V_{R_{W^\perp}}(M_2) = \mathcal{V}_R(M_1 \otimes_Q M_2) \cap W^\perp$$

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Proof uses the fact that $\mathcal{V}_R(M \otimes_R N) = \mathcal{V}_R(M)$ whenever $\operatorname{pd}_R N < \infty$ and $\operatorname{Tor}_i^R(M, N) = 0$ for $i > 0$.

Gives an easy proof of

Theorem

Every cone in k^c is realized by an R-module of finite length.

Theorem

Suppose that $\mathcal{V}_R(M)$ is irreducible and $\mathcal{V}_R(M) \cap W$ is reducible. Then the R_W-syzygies of M split.

Uses a theorem of Bergh 2007: if M is MCM, then $\mathcal{V}_R(M)$ is irreducible if M is indecomposable.

THANK YOU!