

Serial Categories and an Infinite Pure Semisimplicity Conjecture

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- Finite representation type and pure semisimplicity.
- (Uni)Serial rings and modules
- Locally finite modules and categories; the coalgebra formalism.
- (Infinite) Representations of locally serial algebras
- Left vs Right locally finite pure semisimplicity for infinite dim. algebras

Pure Semisimplicity Conjecture

Theorem (Auslander, Fuller, Reiten, Ringel, Tachikawa)

If R is a ring of finite representation type, i.e. there are only finitely many isomorphism types of indecomposable left (equivalently, right) modules, then every left and every right module decomposes as a \bigoplus of indecomposable modules.

*Conversely, if every left **and** every right module decomposes as a \bigoplus of indecomposable modules, then R is of finite representation type.*

Question: when complete decomposition into indecomposables on one side (pure semisimplicity) implies fin. rep. type? Auslander: true for Artin algebras.

(Pure Semisimplicity Conjecture)

If in the category of left R -modules every object decomposes as a \bigoplus of indecomposables, then R is of finite representation type.

Ex: a serial artinian ring is of fin. rep. type. Every module decomposes as a direct sum of uniserial modules (simplest non-semisimple example?).

An extension

For a finite dimensional algebra, the category of left A -modules has an intrinsic finiteness property: every module is the sum of its finite dimensional submodules (i.e. it is locally finite). Equivalently, it is generated by modules of finite length, which are also finite dimensional.

A category which is abelian (or Grothendieck) and is generated by objects of finite length is called finitely accessible. [Gruson-Jensen]

A natural extension is to look at the category of locally finite modules over an arbitrary algebra. If A is finite dimensional, this category can be considered “finite”, while if A is arbitrary, it is only “locally finite”.

Locally finite categories

More generally, one can consider a finitely accessible \mathbb{K} -linear category with where simple modules are finite dimensional. I.e. objects are locally finite. Such categories have also been called of finite type.

- $\text{lf.}A\text{-Mod} =$ locally finite A -modules over an algebra A .

Not every locally finite linear category is of this type. Example: the category of nilpotent matrices $A : V \rightarrow V$, with natural morphisms $f : (V, A) \rightarrow (V', A')$, $A'f = fA$.

- More generally, the locally nilpotent representations of any quiver Q . This is the subcategory of $\text{Rep}(Q)$ consisting of modules M such that each $x \in M$ is annihilated by a cofinite monomial ideal of $\mathbb{K}[Q]$ - the path algebra of Q .
- Similarly, locally nilpotent modules over a monomial algebra of a quiver Q .
- Rational modules over an algebraic group (scheme) G .
- (Co)chain Complexes of vector spaces, or more generally, of locally finite modules.

Locally finite linear categories and comodules

In general, by a result of Gabriel, a finitely accessible category is in duality with the category of pseudocompact modules over a pseudocompact ring.

In particular, by a remark of Takeuchi, a finitely accessible linear category (i.e. locally finite, or of finite type, in alternate terminology) is the category of comodules $\text{Comod-}C$ over a coalgebra C .

What is a coalgebra?

K -algebra A

$$m : A \otimes A \rightarrow A \quad \& \quad u : K \rightarrow A$$

K -coalgebra C

$$\Delta : C \rightarrow C \otimes C \quad \& \quad \varepsilon : C \rightarrow K$$

With appropriate compatibility conditions, dual to the associativity and unital axioms.

Locally finite categories and comodules: Examples

- Rational modules over an algebraic group scheme G = comodules over the function (Hopf) algebra of G .
- Locally nilpotent representations of a quiver Q = comodules over the quiver coalgebra of Q . If Q has no oriented cycles and finitely many arrows between any two vertices, then this category is equivalent to locally finite modules over the path algebra of Q , and over the *complete path algebra* of Q [Dascalescu,I,Nastasescu]
- $\text{lf-}A\text{-Mod}$ = right comodules over the coalgebra of representative functions $R(A) = \{f \in A^* \mid \ker(f) \text{ contains a cofinite ideal}\}$.

[quiver coalgebra of Q = vector space $\mathbb{K}Q$ spanned by Q with comultiplication Δ and counit ε defined on the basis of paths as

$$\Delta(p) = \sum_{p=qr} q \otimes r$$

$$\varepsilon(p) = \delta_{\text{length}(p),0}$$

Infinite Version of the Pure Semisimplicity Conjecture

(Infinite Pure Semisimplicity Conjecture)

Let A be an algebra. If in the category of locally finite left A -modules every module decomposes as a \bigoplus of indecomposable modules, does the same follow for locally finite right A -modules?

More generally, the problem can be formulated in the language of coalgebras and left and right comodules.

Theorem (Simson)

If every left comodule decomposes as a direct sum of finite dimensional comodules, then it is not necessary that every right comodule decomposes as a \bigoplus of finite dimensional comodules. Example: comodules over the path coalgebra of

$$\mathbb{A}_\infty : \bullet \rightarrow \bullet \rightarrow \dots \rightarrow \bullet \rightarrow \bullet \rightarrow \dots$$

Serial categories

To study this problem, it is natural to look at the simplest situation beyond semisimple - serial.

Definition

We call a locally finite category serial if every finite dimensional object is serial.

We call a locally finite category weak serial if every injective (indecomposable) object is (uni)serial

A - f.d. algebra is left serial if the projective left modules (equivalently, injective right modules) are serial. A is serial (left and right) if every module is serial ($= \bigoplus$ uniserial modules).

Coalgebra formalism: $\mathcal{B} = \text{comod} - C$ is Grothendieck, so we work with injectives. C is right serial if injectives are serial. C is left and right serial if and only if f.d. C -comodules are serial [Cuadra, J.Gomez-Torrecillas].

In particular, the definitions make sense for an algebra A and the categories of left and/or right locally finite modules.

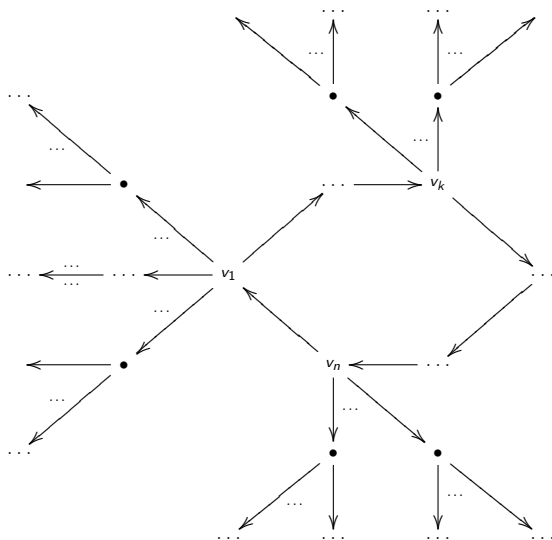
Question [Cuadra, J. Gomez-Torrecillas]: does every object in such a serial category decompose as \bigoplus of serials?

Theorem

Every “weakly serial category” over an algebraically closed field is equivalent to a category of locally nilpotent representations (i.e. left comodules of the quiver coalgebra) over a monomial algebra of a quiver of the following type (“cycle-tree”).

In particular, one obtains the classification of finite dimensional one-sided serial algebras.

Cycle-Tree



Locally finite categories

Think of locally finite left A -modules. Any locally finite category is fully embedded in a category of modules: $R - \text{Mod}$ over the complete algebra $R = C^*$. The corresponding pre-torsion functor is denoted Rat . So $Rat(M) = \text{sum of subobjects of } M \text{ which belong to } \mathcal{B}$. We let $J = Jac(B)$.

Example: if \mathcal{B} is the category of locally nilpotent representations of the quiver Q , then we have full embeddings:

$$\mathcal{B} \hookrightarrow \text{Rep}(Q) \hookrightarrow R - \text{mod}$$

where $R = \overline{\mathbb{K}[Q]}$ is the complete path algebra of Q . It can be defined as the completion of the path algebra $\mathbb{K}[Q]$ with respect to the topology of cofinite monomial ideals, and explicitly as

$$R = (\alpha_p)_{p\text{-path}}; \alpha_p \in \mathbb{K}$$

with multiplication given by “convolution”

$$(\alpha_p)_p * (\beta_q)_q = \left(\sum_{r=pq} \alpha_p \beta_q \right)_r$$

Serial Categories and primary abelian groups

Locally finite category \mathcal{B} (for example, locally finite A -modules), which is serial. \mathcal{B} embeds in $R\text{-Mod}$, R -complete, $J = \text{Jac}(R)$.

$\text{Rat} : \mathcal{B} \rightarrow R\text{-Mod}$, $\text{Rat}(M)$ = the largest submodule of M which belongs to \mathcal{B} .

Definition

Let $M \in \mathcal{B}$ and N be a submodule (subobject) of M . Then N will be called basic if:

- (i) $N \cap J^n M = J^n N$ for all $n > 0$.
- (ii) $N + JM = M$, equivalently, $J(M/N) = M/N$.
- (iii) N decomposes as a direct sum of finite dimensional uniserials.

This is an adaptation of the notion of basic subgroup; Recall that for a p -primary abelian groups M , a subgroup N is basic if N is a direct sum of cyclic groups, M/N is injective and N is *servng*.

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Example

Assume \mathcal{B} has an infinite dimensional indecomposable injective object E . Let $M_E = \text{Rat}(\prod_n E_n)$, where $E_n = L_n(E)$ is the submodule of length n of E . Then $\bigoplus_n E_n$ is a basic submodule of M .

Proposition

Any two basic submodules of $M \in \mathcal{B}$ are isomorphic.

Consequently, if M_E above is serial, one shows that it would be its own basic submodule, so isomorphic to $\bigoplus_n E_n$. This is impossible.

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Corollary

If there is an infinite dimensional indecomposable injective, then not all modules in \mathcal{B} are serial. This answers the original question of [Cuadra, J. Gomez-Torrecillas].

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- Define: analogue of height in abelian groups. It seems more appropriate to be called depth here (?). If N is a uniserial submodule of $M \in \mathcal{B}$, then denote $\text{depth}_M(N)$ the sup of $\text{length}(L/N)$ for all uniserial extensions L of N . For an abelian group M and an element $y \in M$, its height is the maximum n such that the equation $p^n x = y$ has a solution, or infinity if such max DNE.
- Serving subobject N of $M \in \mathcal{B}$: if $\text{depth}_N(X) = \text{depth}_M(X)$ for all X .

Serial Categories and primary abelian groups

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One can then prove:

Theorem

Kulikov's Criterion for serial categories. $M \in \mathcal{B}$ is \bigoplus of f.d. uniserials iff $M = \bigcup M^{(n)}$ such that the simple submodules of $M^{(n)}$ have bounded depth in M .

Prufer Theorem 1. $M \in \mathcal{B}$. If M has finite Loewy length, then M is a \bigoplus of f.d. uniserials.

Prufer Theorem 2. $M \in \mathcal{B}$ is countable dimensional, and simple submodules have finite depth, then M is a \bigoplus of f.d. uniserials.

$M \in \mathcal{B}$ and all simple submodules have infinite depth, then M is \bigoplus of infinite dim. injective indec's.

Serial Categories and primary abelian groups

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Theorem (Indecomposables of serial categories)

The indecomposable objects of \mathcal{B} are either finite dimensional uniserial, or countable dimensional serial and injective.

Corollary

If there are infinite dimensional indecomposable simple objects in \mathcal{B} , then not every object decomposes into \bigoplus of indecomposable objects.

Serial Categories, and Pure Semisimplicity

Locally finite category \mathcal{B} (for example, locally finite A -modules), which is serial. \mathcal{B} embeds in $R\text{-Mod}$, R -complete, $J = \text{Jac}(R)$.

$\text{Rat} : \mathcal{B} \rightarrow R\text{-Mod}$, $\text{Rat}(M)$ = the largest submodule of M which belongs to \mathcal{B} .

Consider the half line quiver

$$\mathbb{A}_\infty : \bullet \rightarrow \bullet \rightarrow \dots \rightarrow \bullet \rightarrow \bullet \rightarrow \dots$$

Let A be its path algebra, or its complete path algebra.

Theorem

The category of locally finite left A -modules is equivalent to the category of locally nilpotent representations of ${}_\infty\mathbb{A}$, and every object decomposes as a direct sum of indecomposable (even uniserial) objects. The category of locally finite right A -modules is equivalent to the category of locally nilpotent representations of \mathbb{A}_∞ , and not every object in this category decomposes into direct sum of indecomposable objects.

The path algebra $\mathbb{K}[A_\infty]$ of A_∞ embeds in the complete path algebra $\overline{\mathbb{K}[A_\infty]}$.

Theorem

Let A be an algebra such that $\mathbb{K}[A_\infty] \subseteq A \subseteq \overline{\mathbb{K}[A_\infty]}$. Then:

- (i) A is finitely serial, i.e. finite dimensional representations of A are all serial.
- (ii) Every locally finite left A -module is a direct sum of indecomposable (even finite dimensional uniserial) modules.
- (iii) Not every locally finite right A -module is a direct sum of indecomposable.

Summary

A -algebra. $\text{lf-}A\text{-mod}$ - locally finite A -modules. Adapting techniques of primary abelian groups, we get:

- We classify the categories which occur as categories of locally finite A -modules and are serial or weak serial, and finite dimensional serial or one-sided serial algebras.
- For an algebra which is finitely serial (f.d. representations are serial):
 - we find criteria for when an object is a direct sum of uniserials;
 - we find the indecomposable locally finite left (and right) A -modules;
 - we show that if there are infinite dimensional injective indecomposables, then there are locally finite modules which don't decompose as \bigoplus of uniserials;
 - we find classes of algebras for which all left lf modules are \bigoplus of indecomposables, and every left lf module is serial, but not every right lf module is serial, and there are lf right modules which are not \bigoplus of indecomposables.

1. (with S.Dascalescu, C.Nastasescu) *Quiver algebras, path coalgebras, and coreflexivity*, (25p), preprint arXiv, **Pacific Journal Mathematics**, 2013.
2. *Serial categories, quantum groups and the pure semisimplicity conjecture*, (34p) **preprint**, preprint dornsife.usc.edu/mci



T H A N K Y O U !