Diophantine Supports of Coherent Functors

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(Joint with S. L'Innocente)

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The special linear algebra

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- Thus $L = kE \oplus kF \oplus kH$, where

$$E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \text{and} \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The affine plane

- The Lie algebra L acts on the **affine** k-**plane** $k[x, y]$ by derivations:

$$E = x\frac{\partial}{\partial y}, \quad F = y\frac{\partial}{\partial x}, \quad \text{and} \quad H = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}.$$

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- $E(x^i y^j) = jx^{i+1}y^{i-1}, \quad F(x^i y^j) = ix^{i-1}y^{i+1}, \quad H(x^i y^j) = (i-j)x^i y^j.$

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- The affine plane decomposes as a direct sum

$$k[x, y] = \oplus_{n \geq 0} k[x, y]_n = \oplus_{n \geq 0} L(n),$$

where $k[x, y]_n = L(n)$ is the $(n+1)$-dimensional space of homogeneous polynomials of total degree $i + j = n$.

The universal enveloping algebra

Definition

The **universal enveloping algebra** $U(L)$ of L is the k-algebra

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- We are interested in the function $n \mapsto \dim_k \mathrm{Ker}\ r(n)$ on the natural numbers $\mathbb{N}$ and the **support**

$$\mathrm{Supp}(r) = \{n \in \mathbb{N} : \mathrm{Ker}\ r(n) \neq 0\}.$$

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- If $M = U(L)/U(L)r$, then $\operatorname{Ker} r(n) \cong \operatorname{Hom}_{U(L)}(M, L(n))$.

Example (The Casimir element)

If $r = Q = 2EF + H^2 + 2FE$, then $r(n) = (n^2 + 2n)I_{n+1}$, and

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Example (H)

Recall that $H(x^{n-i}y^i) = (n-2i)x^{n-i}y^i$. Therefore

$$\mathrm{Supp}(H) = \{2n : n \in \mathbb{N}\} \quad \text{and} \quad \mathrm{Supp}(H-1) = \{2n+1 : n \in \mathbb{N}\}.$$

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Example

$$\mathrm{Supp}(Q - (H^4 + 2H^2)) = \{n^2 : n \in \mathbb{N}\}$$

The Pell equation

Example (L'Innocente and Macintyre)

Let $D > 0$ be a square-free integer and $r = Q - DH^2$. Then

$$r(n)(x^{n-i}y^i) = [n^2 + 2n - D(n-2i)^2](x^{n-i}y^i).$$

So $n \in \mathrm{Supp}(r)$ iff $\exists i$, $0 \leq i \leq n$, such that

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- If a pair (u, v) of integers is a solution to the **Pell Equation**

$$u^2 - Dv^2 = 1,$$

then $|v| < |u|$ and $u + v \equiv 1 \pmod 2$, so that

$$\mathrm{Supp}(r) = \{n : \exists v\ [(n+1)^2 - Dv^2 = 1]\}.$$

The $\mathbb{Z}$-grading of $U(L)$

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- Every $r \in U(L)$ has a unique representation as

$$r = \sum_{m<0} F^{-m} p_m(Q, H) + p_0(Q, H) + \sum_{m>0} E^m p_m(Q, H),$$

where $p_m(u, v) \in k[u, v],\ m \in \mathbb{Z}$.

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Theorem (IH)

Given $r \in U(L)$, there is a b such that for all $n \in \mathbb{N}$, $\dim_k(Ker\ r(n)) \leq b$.

A computable function

- The representation $L(n)$ is an $(n+1)$-dimensional k-vector subspace with basis $\mathcal{B}_n := \{x^n, x^{n-1}y, \ldots, xy^{n-1}, y^n\}$.

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$$r(n)(i,j) = \begin{cases} \frac{(n-j)!}{(n-i)!} p_{j-i}(n^2+2n, n-2j), & \text{if } i \geq j; \\ \frac{j!}{i!} p_{j-i}(n^2+2n, n-2j), & \text{if } i \leq j. \end{cases}$$

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- The function $n \mapsto \operatorname{rk} r(n)$ is **computable:** $\operatorname{rk} r(n)$ is the least p such that every $p \times p$ minor of $r(n)$ vanishes. Thus the function $n \mapsto \dim_k \operatorname{Ker} r(n) = n - \operatorname{rk} r(n)$ is computable.

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Theorem (IH and S. L'Innocente)

If $C \in fp(U(L)\text{-mod}, Ab)$ is a coherent functor, then the function $n \mapsto \dim_k C(L(n))$ is computable.

Diophantine subsets of $\mathbb{Z}$

Definition

A subset $D \subseteq \mathbb{Z}$ is **Diophantine** if there is a polynomial $p(u, \overline{v})$ with integer coefficients such that $d \in D$ iff there is a tuple $\overline{z}$ of integers such $p(d, \overline{a}) = 0$.

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Theorem (Matijašević-Robinson-Putnam-Davis)

A subset $D \subseteq \mathbb{Z}$ is Diophantine and co-Diophantine iff there is an effective procedure to decide if $n \in D$.

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A subset $D \subseteq \mathbb{Z}$ is Diophantine and co-Diophantine iff there is an effective procedure to decide if $n \in D$.

Corollary

If C is a coherent functor, then $Supp(C) := \{n \in \mathbb{N} : \ C(L(n)) \neq 0\}$ is Diophantine and co-Diophantine.

The Ziegler spectrum of $k[x, y]$

Definition (Ziegler)

The (left) **Ziegler spectrum** of a ring R is the topological space $\mathrm{Zg}(R)$ whose points X are the algebraically compact indecomposable (left) R-modules, where a basis of quasi-compact open subsets is given by the supports

$$\mathcal{O}(C) = \{X \in \mathrm{Zg}(R) : C(X) \neq 0\}$$

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Theorem (Prest)

If Λ is an artin algebra, then the isolated points of the Ziegler spectrum $Zg(\Lambda)$ are the indecomposable representations of finite length. These points form a dense, discrete, and open subset.

The localization of $U(L)$ at $k[x, y]$

Theorem (IH)

Let $\mathcal{S} = \mathcal{S}(k[x, y]) \subseteq fp(U(L)\text{-}mod, Ab)$ be the Serre subcategory of coherent functors that vanish on $k[x, y]$. The localization $fp(U(L)\text{-}mod, Ab)/\mathcal{S}$ has homological dimension 0.

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- Equivalently, there is a ring epimorphism $U(L) \to U'(L)$ with $U'(L)$ von Neumann regular that induced the equivalence

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- The epimorphism induces a homeomorphism between $\mathrm{Zg}(U'(L))$ and the closure in $\mathrm{Zg}(U(L))$ of the points $\{L(n) : n \in \mathbb{N}\}$.

Arithmetic progressions

Example (IH)

Given a, $b \in \mathbb{N}$, there is a coherent functor $C \in \mathrm{fp}(U(L)\text{-mod}, \mathrm{Ab})$ s.t.

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Corollary

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- If $E \in \mathrm{Zg}(U'(L))$, and $I = \mathrm{ann}[\mathrm{soc}(E)]$, then the composition

$$U(L) \to U'(L) \to U'(L)/I$$

is a ring epimorphism into a hereditary von Neumann regular prime ring.

References

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