Counting Moduli of Quiver Representations with Relations

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Quiver with Relations

Fix a set R of homogeneous elements in kQ with respect to the bigrading: $kQ = \bigoplus_{v,w \in Q_0} vkQw$. If $M(r) = 0$ for all $r \in R$, then we say M is a *representation of Q with relations R*. The *path algebra of Q with relations R* is the quotient algebra $A := kQ/\langle R \rangle$. A representation of Q with relations R naturally becomes an A-module.

The assignment $M \mapsto M(r)$ defines a polynomial map $ev(r) : \mathrm{Rep}_\alpha(Q) \to \mathrm{Hom}(k^{\alpha(tr)}, k^{\alpha(hr)})$, which is represented by an $\alpha(hr) \times \alpha(tr)$ matrix with entries in $k[\mathrm{Rep}_\alpha(Q)]$. Let $\tilde{R} \subseteq k[\mathrm{Rep}_\alpha(Q)]$ be the ideal generated by the entries of all $ev(r)$ for which $r \in R$. The representation space $\mathrm{Rep}_\alpha(A)$ is the scheme $\mathrm{Spec}(k[\mathrm{Rep}_\alpha(Q)]/\tilde{R})$.

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One-point Extensions from Quivers

Let E be a representation of Q. The *one-point extension* of Q by E is the triangular algebra $kQ[E] := \left(\begin{smallmatrix} kQ & 0 \\ E & k \end{smallmatrix}\right)$.

Suppose that E is presented by $0 \to P_1 \xrightarrow{D} P_0 \to E \to 0$ with $P_1 = \oplus_v b_v^1 P_v$ and $P_0 = \oplus_v b_v^0 P_v$. Then the algebra $A = kQ[E]$ can be presented by a new quiver $Q(E)$, which is obtained from Q by adjoining a new vertex "$-$" and for each P_v in P_0 a new arrow from "$-$" to the vertex v. The relations are clearly given by the matrix D.

The one-point coextension $kQ^\circ[E] := \left(\begin{smallmatrix} k & 0 \\ E^* & kQ \end{smallmatrix}\right)$ can be similarly described using injective presentation of E. By convention, the newly adjoined vertex is denoted by "$+$".

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Examples of One-point Extensions

Consider the quiver $1 \xrightarrow{a} 2 \xrightarrow{b} 3$ with relation $ab = 0$. The corresponding algebra A is one-point-(co)extended from the Dynkin quiver A_2 by the simple S_2, because $0 \to P_3 \xrightarrow{b} P_2 \to S_2 \to 0$ and $0 \to S_2 \to I_2 \xrightarrow{a} I_1 \to 0$.

Consider quiver $1 \xrightarrow{a,b,c} 2 \xrightarrow{x,y,z} 3$ with relation $AX = 0$, where $A = \begin{pmatrix} c & 0 & -a \\ -b & a & 0 \\ 0 & -c & b \end{pmatrix}$, $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$. It is coextended from K_3 by E, where $0 \to E \to 3I_2 \xrightarrow{A} 3I_1 \to 0$. Note that E is *not* general in $\mathrm{Rep}_{(3,3)}(K_3)$. (Beilinson's $\mathbb{P}^2$)

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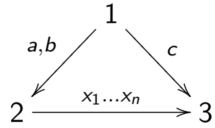
Consider quiver 

with relation $AX = 0$, where $A = (a, b)$ and $X = \begin{pmatrix} x_1 & x_2 & \cdots & x_{n-1} \\ x_2 & x_3 & \cdots & x_n \end{pmatrix}$. Then it is extended from K_n by $E_n \oplus P_3$, or coextended from K_2 by $E_n^\circ \oplus I_1$. Here,

$$0 \to (n-1)P_3 \xrightarrow{X^T} 2P_2 \to E_n \to 0,$$

$$0 \to E_n^\circ \to nI_2 \xrightarrow{B^T} (n-1)I_1 \to 0,$$

where $B = \begin{pmatrix} a & b & 0 & 0 & \cdots & 0 \\ 0 & a & b & 0 & \cdots & 0 \\ \vdots & \vdots & & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & a & b \end{pmatrix}$. Note that E° is *exceptional*, but E is not.

Affine Representation Varieties

The dimension vector for $\text{Rep}(Q[E])$ consists of two parts: the dimension of the vector space supported on "$-$" and outside "$-$". To simplify notation, a dimension vector with tilde, say $\tilde{\alpha}$, consists of two components (α_-, α), or (α, α_+) for coextension.

$\text{Rep}_{(n,\alpha)}(Q[E])$ is the subvariety of $\text{Rep}_\alpha(Q) \times \text{Hom}(nE, k^\alpha)$

$$\{(M, f) \in \text{Rep}_\alpha(Q) \times \text{Hom}(nE, k^\alpha) \mid f \in \text{Hom}_Q(nE, M)\}.$$

$\text{Rep}_{(\alpha,n)}(Q^\circ[E])$ is the subvariety of $\text{Rep}_\alpha(Q) \times \text{Hom}(k^\alpha, nE)$

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Counting Affine

For any dimension vector β, we define $\operatorname{Hom}_Q(E,\alpha)_\beta =$

$$\{(M,\phi,E_1,M_1) \in \operatorname{Rep}_\alpha(Q) \times \operatorname{Hom}(E,k^\alpha) \times \operatorname{Gr}^\beta(E) \times \operatorname{Gr}_\beta(\alpha) \mid \phi \in \operatorname{Hom}_Q(E,M), E/\operatorname{Ker}\phi = E_1, \operatorname{Im}\phi = M_1\}.$$

Lemma

$p : \operatorname{Hom}_Q(E,\alpha)_\beta \to \operatorname{Gr}^\beta(E) \times \operatorname{Gr}_\beta(\alpha)$ is a fibre bundle with fibre

$$\operatorname{GL}_\beta \times \operatorname{Rep}_{\alpha-\beta}(Q) \times \bigoplus_{a\in Q_1}(\operatorname{Hom}(k^{(\alpha-\beta)(ta)}, k^{\beta(ha)}).$$

So $$r_{(n,\alpha)}(Q[E]) := \sum_{\alpha=\gamma+\beta} \frac{|\operatorname{Gr}^\beta(nE)|}{\langle\gamma,\beta\rangle|\operatorname{GL}_n|} r_\gamma(Q).$$

Dually $$r_{(\alpha,n)}(Q^\circ[E]) = \sum_{\alpha=\gamma+\beta} \frac{|\operatorname{Gr}_\gamma(nE)|}{\langle\gamma,\beta\rangle|\operatorname{GL}_n|} r_\beta(Q).$$

A *slope function* μ is certain quotient of two linear functionals σ/θ on $\mathbb{Z}^{Q_0}$ with $\theta(\alpha) > 0$ for any dimension vector α.

Definition. A representation M is called μ-semi-stable (resp. μ-stable) if $\mu(\overline{L}) \leqslant \mu(\overline{M})$ (resp. $\mu(\overline{L}) < \mu(\overline{M})$) for every non-trivial subrepresentation $L \subset M$. Let $\text{Rep}_\alpha^\mu(Q)$ be the variety of α-dimensional μ-semistable representations.

Facts. There is a *good categorical quotient* $q : \text{Rep}_\alpha^\mu(Q) \to \text{Mod}_\alpha^\mu(Q)$, and its restriction to the stable representations is a *geometric quotient*.

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Harder-Narasimhan Filtration

HN filtration. Fix a slope function μ. Every representation M has a unique filtration

$$0 = M_0 \subset M_1 \subset \cdots \subset M_{m-1} \subset M_m = M$$

such that $\begin{cases} N_i = M_i/M_{i+1} \text{ is } \mu\text{-semi-stable}, \\ \mu(\overline{N}_i) > \mu(\overline{N}_{i+1}). \end{cases}$

Key Lemma

$$|\operatorname{Rep}_\alpha^\mu(A)| = \sum_* (-1)^{s-1} |\operatorname{Frep}_{\alpha_1 \cdots \alpha_s}(A)|,$$

where the sum runs over all decomposition $\alpha_1 + \cdots + \alpha_s = \alpha$ of α into non-zero dimension vectors such that $\mu(\sum_{l=1}^k \alpha_l) < \mu(\alpha)$ for $k < s$.

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For any decomposition of dimension vector $\alpha = \sum_{i=1}^{s} \alpha_i$, we define $\mathrm{Fl}_{\alpha_s \cdots \alpha_1} := \prod_{v \in Q_0} \mathrm{Fl}_{\alpha_s(v) \cdots \alpha_1(v)}$, where $\mathrm{Fl}_{\alpha_s(v) \cdots \alpha_1(v)}$ is the usual flag variety parameterizing flags of subspaces of dimension $\alpha_1(v) < \dot{\alpha}_2(v) < \cdots < \dot{\alpha}_{s-1}(v)$ in $k^{\alpha(v)}$. To simplify the notation, we denote $\dot{\alpha}_i := \sum_{j=1}^{i} \alpha_j$.

Definition

We define the *Frep* variety $\mathrm{Frep}_{\alpha_s \cdots \alpha_1}(A)$

$$= \{(M, L_1, \ldots, L_{s-1}) \in \mathrm{Rep}_\alpha(A) \times \mathrm{Fl}_{\alpha_s \cdots \alpha_1} \mid L_1 \subset \cdots \subset L_s = M\}.$$

Consequence of Weil Conjecture

Lemma

If X is counted by a rational function $P_X \in \mathbb{C}(t)$, then it must lie in $\mathbb{Z}[t]$. Moreover if X is l-pure, then

$$P_X(q) = \sum_{i \geq 0} \dim H_c^{2i}(X, \mathbb{Q}_l) q^i,$$

is the l-adic Poincaré polynomial

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F-polynomial Count

Definition

We say an algebra A is *polynomial-count* if each $\mathrm{Rep}_\alpha(A)$ is polynomial-count. It is called *F-polynomial-count* if each $\mathrm{Frep}_{\alpha_1\cdots\alpha_s}(A)$ is polynomial-count.

In particular, if A is F-polynomial-count, then each $\mathrm{Mod}_\alpha^\mu(A)$ is polynomial-count when it is a geometric quotient. We conjecture that the assumption of being a geometric quotient can be dropped. Moreover, we do not know a single example where A is polynomial-count but not F-polynomial-count. We conjecture that if each $\mathrm{Frep}_{\alpha_1\alpha_2}(A)$ is polynomial-count, then A is F-polynomial-count.

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Counting 2-step Frep of $Q[E]$

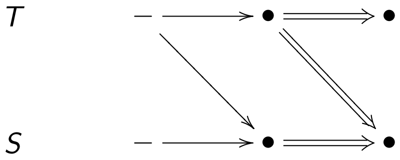
Lemma

$p : \mathrm{Frep}_{\tilde{\beta},\tilde{\gamma}}(Q[E]) \to \mathrm{Fl}_{\tilde{\beta},\tilde{\gamma}}$ is a fibre bundle with fibre

$$\mathrm{Rep}_{(\alpha_-,\gamma)}(Q[E]) \times \mathrm{Rep}_{\tilde{\beta}}(Q[E]) \times \prod_{a\in Q_1} \mathrm{Hom}(k^{\beta(ta)}, k^{\gamma(ha)})$$

So $\quad r_{\tilde{\beta},\tilde{\gamma}}(Q[E]) = \langle \beta, \gamma \rangle^{-1} \left[\begin{smallmatrix} \alpha_- \\ \gamma_- \end{smallmatrix} \right] |\,\mathrm{GL}_{\beta_-}| r_{\tilde{\beta}}(Q[E]) r_{(\alpha_-,\gamma)}(Q[E]).$

Proof.



Counting Frep of $Q[E]$

The 2-step case can be recursively generalized to the n-step case. We only state the analog for the last formula.

$$r_{\tilde{\alpha}_1\cdots\tilde{\alpha}_s}(Q[E]) = \prod_{i=1}^{s} \left[\begin{smallmatrix} \dot{\alpha}_{i,-} \\ \alpha_{i,-} \end{smallmatrix} \right] |\operatorname{GL}_{\dot{\alpha}_{i-1,-}}| r_{(\dot{\alpha}_{i,-},\alpha_i)}(Q[E]). \tag{1}$$

The formula for coextension.

$$r_{\tilde{\alpha}_s\cdots\tilde{\alpha}_1}(Q^\circ[E]) = \prod_{i=2}^{s} \left[\begin{smallmatrix} \dot{\alpha}_{i,+} \\ \alpha_{i,+} \end{smallmatrix} \right] |\operatorname{GL}_{\dot{\alpha}_{i-1,+}}| r_{(\alpha_i,\dot{\alpha}_{i,+})}(Q^\circ[E]).$$

So all Frep varieties can be counted in terms of representation varieties $\operatorname{Rep}_\alpha(Q[E])$.

Counting Moduli

Definition

A representation $E \in \mathrm{Rep}(Q)$ is called *polynomial-count*, if all its Grassmannians $\mathrm{Gr}_\gamma(E)$ are polynomial-count. It is called *add-polynomial-count*, if each nE is polynomial-count.

Theorem

$\mathrm{Rep}_\alpha^\mu(Q[E])$ can be explicitly counted in terms of $\mathrm{Gr}_\gamma(nE)$'s. In particular, if E is add-polynomial-count, then each $\mathrm{Mod}_\alpha^\mu(Q[E])$ is polynomial-count when it is a geometric quotient.

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Counting Quiver Grassmannian

Corollary Assume that $\dim U = \alpha_1$ and $\dim V = \alpha_2$.

$$\sum_{\gamma_1+\gamma_2=\gamma} \langle \gamma_1, \alpha_2 - \gamma_2 \rangle |\operatorname{Gr}_{\gamma_1}(U)||\operatorname{Gr}_{\gamma_2}(V)| = \sum_{[W]} \frac{|\operatorname{Ext}_Q(U,V)_W|}{|\operatorname{Ext}_Q(U,V)|} |\operatorname{Gr}_\gamma(W)|.$$

Now suppose that $\operatorname{Ext}_Q(U,V) = 0$. Then

$$F_\bullet(U \oplus V) = \sum_{\gamma_1,\gamma_2} \langle \gamma_1, \alpha_2 - \gamma_2 \rangle \operatorname{Gr}_{\gamma_1}(U) \operatorname{Gr}_{\gamma_2}(V) x^{\gamma_1+\gamma_2}.$$

Hence,if both U and V are (add)-polynomial-count, then so is $U \oplus V$. Moreover, if $\operatorname{Ext}_Q(V,U) = k^e$ and W is the only non-trivial middle term of the extensions, then

$$(q^e - 1)F_\bullet(W) = q^e \sum_{\gamma_1,\gamma_2} \langle \gamma_2, \alpha_1 - \gamma_1 \rangle \operatorname{Gr}_{\gamma_2}(V) \operatorname{Gr}_{\gamma_1}(U) x^{\gamma_1+\gamma_2} - F_\bullet(U \oplus V).$$

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Acyclic Cluster Theory

For any indecomposable rigid T of an acyclic quiver, *Cluster theory* allows us recursively use the last formula to compute all $\mathrm{Gr}_\gamma(T)$'s. Each step of recursion is related to the *cluster mutation*.

In particular, *Positivity conjecture* is true in acyclic cases: each $\mathrm{Gr}_\gamma(T)$ is counted by a positive polynomial, because $\mathrm{Gr}_\gamma(T)$ is smooth and thus l-pure.

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Example 1 continued

Come back to Beilinson's $\mathbb{P}^2$ (coextended from K_3 by E). It is known that for a general representation E_g of dimension $(6,3)$, $\mathrm{Gr}_{(1,1)}(E_g)$ is an elliptic curve. So E_g is *not* polynomial-count. However, for this special E, $\mathrm{Gr}_{(1,1)}(E)$ is three $\mathbb{P}_1$'s intersecting at a point. With a little effort one can show that E is actually polynomial-count.

Let $A := kK_m^\circ[E]$ be the algebra coextended from K_m by a representation E of dimension ϵ. For any dimension vector $\alpha = (\alpha_1, \alpha_2)$ of K_m, there is a unique choice of weight σ up to scalar such that $\sigma(\alpha) = 0$. There are two ways to extend σ to A. One is $\sigma_+ = (\sigma, -\delta)$, and the other is $\sigma_- = (\delta, \sigma)$, for some sufficiently small positive number δ.

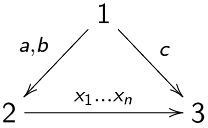
Proposition

$$\mathrm{Mod}^{\mu_+}_{(\gamma,1)}(A) \cong \mathrm{Gr}_\gamma(E) \text{ and } \mathrm{Mod}^{\mu_-}_{(\gamma_1,1,1)}(A) \cong \mathrm{Gr}_{(\gamma_1,1)}(E),$$
$$|\,\mathrm{Mod}^{\mu_-}_{(1,2,1)}(A)| = |\,\mathrm{Gr}_{(1,2)}(E)| + ([m-1] - [\epsilon_2 - 1])|\,\mathrm{Gr}_{(1,1)}(E)|,$$
$$|\,\mathrm{Mod}^{\mu_-}_{(2,2,1)}(A)| = |\,\mathrm{Gr}_{(2,2)}(E)| + ([2m-1] - [\epsilon_2 - 1])|\,\mathrm{Gr}_{(2,1)}(E)|,$$
$$\cdots\cdots,$$

where $[n]$ is the quantum number.

Example 2 continued

(coextended from K_2 by $E_n^\circ \oplus I_1$)



$$[C.Szántó] \quad |\operatorname{Gr}_\gamma(E_n^\circ)| = \begin{cases} 1 & \gamma = (0,0), (n+1, n) \\ \left[\begin{smallmatrix} n-\gamma_1 \\ \gamma_2-\gamma_1 \end{smallmatrix} \right] \left[\begin{smallmatrix} \gamma_2+1 \\ \gamma_1 \end{smallmatrix} \right] & \text{otherwise,} \end{cases}$$

where $\left[\begin{smallmatrix} n \\ m \end{smallmatrix} \right]$ is the quantum binomial coefficient. So we are able to find all $|\operatorname{Mod}_\alpha^\mu(A_n)|$. For example,

$$|\operatorname{Mod}_{(1,1,1)}^{\mu-}(A_n)| = q^2 + 2q + 1,$$
$$|\operatorname{Mod}_{(1,1,1)}^{\mu+}(A_n)| = [n] + [3] - 1,$$
$$|\operatorname{Mod}_{(2,2,1)}^{\mu-}(A_n)| = q^4 + 2q^3 + 4q^2 + 2q + 1.$$

However, all $\operatorname{Mod}_{(1,1,1)}^{\mu-}(A_n)$ are different, they are Hirzebruch surfaces $\mathbb{F}_n$.

Example 3

Consider quiver $1 \xrightarrow{a,b,c} 2 \xrightarrow{x,y,z} 3$ with relation $xa + yb + zc = 0$. It is coextended from K_3 by a rigid module presented by $0 \to E \to 3I_2 \xrightarrow{(a\ b\ c)} I_1 \to 0$. Similar calculation as before gives

$$
\begin{aligned}
&|\operatorname{Mod}^{\mu\pm}_{(1,1,1)}(A)| = [2][3],\\
&|\operatorname{Mod}^{\mu\pm}_{(2,1,1)}(A)| = |\operatorname{Mod}^{\mu\pm}_{(1,1,2)}(A)| = [3],\\
&|\operatorname{Mod}^{\mu\pm}_{(1,2,1)}(A)| = [3][5],\\
&|\operatorname{Mod}^{\mu-}_{(2,2,1)}(A)| = |\operatorname{Mod}^{\mu-}_{(1,2,2)}(A)| = [3][5](1,0,1),\\
&|\operatorname{Mod}^{\mu-}_{(1,2,2)}(A)| = |\operatorname{Mod}^{\mu+}_{(2,2,1)}(A)| = [3](1,1,3,3,3,1,1).
\end{aligned}
$$

The first one is a divisor $\mathcal{D}$ on $\mathbb{P}^2 \times \mathbb{P}^2$ of bidegree $(1,1)$, or equivalently the complete flag variety $\mathcal{F}_3$ of k^3.

Example 3 - deformed

Now consider the deformation $E' \oplus I_2$ of E, where $0 \to E' \to 2I_2 \xrightarrow{(a\ b)} I_1 \to 0$. Since $\mathrm{Ext}_Q(I_2, E') = k$ with E the only non-trivial middle term, we can compute $F_\bullet(E')$

$$F_\bullet(E') = 1 + [2]x^{(1,0)} + [2]^2 x^{(1,1)} + [2]x^{(2,1)} + x^{(0,2)} + [5]x^{(1,2)} + \left[\begin{smallmatrix}5\\2\end{smallmatrix}\right] x^{(2,2)} + \cdots$$

$$|\operatorname{Mod}^{\mu\pm}_{(1,1,1)}(A)| = (1,3,2,1),$$
$$|\operatorname{Mod}^{\mu\pm}_{(2,1,1)}(A)| = |\operatorname{Mod}^{\mu\pm}_{(1,1,2)}(A)| = (2,2,1),$$
$$|\operatorname{Mod}^{\mu\pm}_{(1,2,1)}(A)| = [3][5],$$
$$|\operatorname{Mod}^{\mu-}_{(2,2,1)}(A)| = |\operatorname{Mod}^{\mu+}_{(1,2,2)}(A)| = [3][5](1,0,1),$$
$$|\operatorname{Mod}^{\mu-}_{(1,2,2)}(A)| = |\operatorname{Mod}^{\mu+}_{(2,2,1)}(A)| = [3](1,1,4,4,3,1,1).$$

Note that the first one is irreducible and singular.

The Δ-Analog

Theorem

If E is add-polynomial-count and $\mathrm{Mod}_\alpha^\mu(Q[E])$ is a geometric quotient, then $\sum_{M\in\mathrm{Mod}_\alpha^\mu(Q[E])}|\mathrm{Gr}_\gamma(M)|$ is polynomial-count for any γ.

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The Construction of the Variety

There are projective varieties related to both the moduli and Grassmannian of representations. We need the tensor product algebra $A_2(A) := kA \otimes kA_2$, where A_2 is the quiver of Dynkin type A_2.

If $\mathrm{Rep}_\alpha^\mu(A)$ contains exclusively μ-stable points, then there is another stability (slope function) $\hat{\mu}$ such that the natural projection $\mathrm{Rep}_{(\gamma,\alpha)}(A_2(A)) \twoheadrightarrow \mathrm{Rep}_\alpha(A)$ induces a surjective map $\mathrm{Mod}_{(\gamma,\alpha)}^{\hat{\mu}}(A_2(A)) \to \mathrm{Mod}_\alpha^\mu(A)$, whose fibre over M is exactly $\mathrm{Gr}_\gamma(M)$.

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Consider

$$\oint(W) = a_W^{-1} \sum_{i=0} (-1)^{i+1} F_i(W) x^{\alpha},$$

where $F_i(W)$ is the number of i-step filtrations of W.
We denote $F_W := \sum_{i=0} (-1)^i F_i(W)$.

Lemma. If W is a direct sum of simples: $\bigoplus_{[S]} S^{m_S}$ and let $q_S = |\operatorname{End}_Q(S)|$, then $F_W = \prod_{[S]} (-1)^{m_S} q_S^{\binom{m_S}{2}}$; otherwise $F_W = 0$.

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The Generating Series

Theorem (Mozgovoy-Reineke)

$$\oint \chi_{\mu_0} = \mathrm{Exp}\left(\frac{A_{\mu_0}(A)}{1-q}\right),$$

$$M_{\mu_0}(A) = \mathrm{Exp}(A_{\mu_0}(A)).$$

Here, Exp *is a* Plethystic Exponential *in the λ-ring* $\mathbb{Q}(q)[[\mathbf{x}]]$ *.*
$A_{\mu_0}(A)$ is the generating series counting equiv. classes of absolutely stable representations with slope μ_0.
$M_{\mu_0}(A)$ is the generating series counting $\mathrm{Mod}_\alpha^\mu(A)$ $(\mu(\alpha) = \mu_0)$*.*

Final Results for $Q[E]$

It follows from the flag generalization of Δ-analog that we are able to compute in terms of $\mathrm{Gr}_\gamma(E)$ the series A_{μ_0} and thus M_{μ_0}.

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The Universal Case $A_2(Q)$

Let us consider a category, which is universal in the sense that it contains all one-point extensions of Q as its full subcategories. It is clearly the module category of $A_2(Q) := kQ \otimes kA_2$.

Let V be an α-dimensional k-vector space. We denote by $\mathrm{In}^{c_d \hookrightarrow c_e}_{c \cap d \hookrightarrow e}(\alpha)$ the incidence variety

$$\{(C, D, E) \in \mathrm{Gr}_c(V) \times \mathrm{Fl}_{e-d,d}(V) \mid \dim(C \cap D) = c_d, \dim(C \cap E) = c_e\},$$

and by $\mathrm{Gr}_d^{b \cap e}(\alpha)$ the incidence variety

$$\{(B, E) \in \mathrm{Gr}^b(V) \times \mathrm{Gr}^e(V) \mid V/B_s = B, V/E_s = E, \dim(B_s \cap E_s) = \alpha - b - e + d\}.$$

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Lemma

$p : \mathrm{Frep}_{(\beta_u,\beta_d),(\gamma_u,\gamma_d)}(A_2(Q)) \to \mathrm{Fl}_{(\beta_u,\beta_d),(\gamma_u,\gamma_d)}$ is a fibre bundle with fibre

$$\bigsqcup_{b,c,d,e,c_d,c_e} \mathrm{In}^{c_d \hookrightarrow c_e}_{c\cap d \hookrightarrow e}(\gamma_d) \times \mathrm{Gr}^{b\cap e}_{e-d}(\beta_u) \times \mathrm{Gr}^{c}(\gamma_u) \times \mathrm{Gr}_{b}(\beta_d) \times \mathrm{GL}_b \times \mathrm{GL}_c \times \mathrm{GL}_e$$

$$\times \prod_{a\in Q_1} \mathrm{Hom}(k^{c_d(ta)}, k^{c_d(ha)}) \times \mathrm{Hom}(k^{(c_e-c_d)(ta)}, k^{c_e(ha)}) \times \mathrm{Hom}(k^{(c-c_e)(ta)}, k^{c(ha)})$$

$$\times \mathrm{Hom}(k^{(d-c_d)(ta)}, k^{d(ha)}) \times \mathrm{Hom}(k^{(e-d-c_e+c_d)(ta)}, k^{e(ha)}) \times \mathrm{Hom}(k^{(\gamma_d-e-c+c_e)(ta)}, k^{\gamma_d(ha)}$$

$$\times \mathrm{Hom}(k^{b(ta)}, k^{(b+d-e)(ta)}) \times \mathrm{Hom}(k^{\beta_u(ta)}, k^{(\beta_u-b-d)(ha)})$$

$$\times \mathrm{Hom}(k^{\gamma_u(ta)}, k^{(\gamma_u-c)(ha)}) \times \mathrm{Hom}(k^{(\beta_d-b)(ta)}, k^{\beta_d(ha)})$$

$$\times \mathrm{Hom}(k^{\beta_u(ta)}, k^{(\gamma_u-c)(ha)}) \times \mathrm{Hom}(k^{(\beta_d-b)(ta)}, k^{\gamma_d(ha)})$$

To be continued...

So $r_{(\beta_u,\beta_d),(\gamma_u,\gamma_d)}(A_2(Q)) := \frac{|\operatorname{Frep}_{(\beta_u,\beta_d),(\gamma_u,\gamma_d)}(A_2(Q))|}{|\operatorname{GL}_{(\alpha_u,\alpha_d)}|}$ is equal to

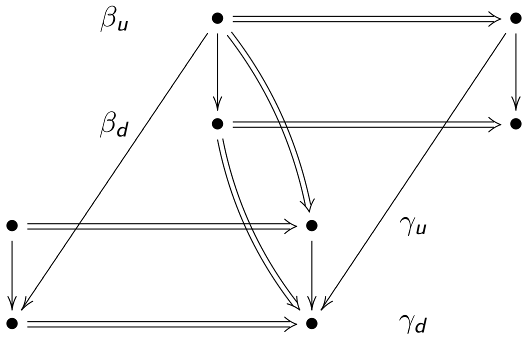
$$\sum_{b,c,e,d_\beta,d_\gamma} t_{(b,c,d,e,c_d,c_e)} \cdot r_{\gamma_u-c} r_{\beta_d-b} \cdot r_{\beta_u-b-d} r_{b+d-e} \cdot$$

$$r_{c_d} r_{c_e-c_d} r_{c-c_e} r_{d-c_d} r_{e-d-c_e+c_d} r_{\gamma_d-c-e+c_e},$$

where $t_{(b,c,d,e,c_d,c_e)} =$
$$\frac{(\langle\beta_u,\gamma_u\rangle\langle\beta_d,\gamma_d\rangle\cdot\langle\beta_d-b,b\rangle\langle c,\gamma_u-c\rangle\cdot\langle e-d,b+d-e\rangle\langle b+d,\beta_u-b-d\rangle)^{-1}\begin{bmatrix} e \\ d \end{bmatrix}}{\langle c-c_e,c_e-c_d\rangle\langle d-c_d,c_d\rangle\langle e-d-c_e+c_d,d+c_e-c_d\rangle\langle\gamma_d-c-e+c_e,c+e-c_e\rangle}.$$

This result can be generalized to the s-step Frep varieties. So we conclude that the algebra $A_2(Q)$ is F-polynomial-count.

Proof by picture



Counting Moduli of $A_2(Q)$

We can do the Δ-analog and S-analog counting for $A_2(Q)$ as well.

Theorem

$\mathrm{Mod}_\alpha^\mu(A_2(Q))$ *has a counting polynomial, which can be explicitly computed.*

An Example

Consider the 3-arrow Kronecker quiver K_3 with dimension vectors $\alpha = (3, 4)$ and $\gamma = (1, 3)$. Let M be a general representation of dimension α, then M has no subrepresentation of dimension $(1, 2)$. So the projection $\mathrm{Gr}_\gamma(M) \to \mathrm{Gr}_1(M_1) \cong \mathbb{P}^2$ is an isomorphism. We find that

$$|\operatorname{Mod}_\alpha^\mu(K_3)| = (1, 0, 1)^2(1, 1, 1, 3, 5, 3, 1, 1, 1),$$
$$|\operatorname{Mod}_{(\gamma,\alpha)}^{\hat{\mu}}(A_2(K_3))| = [3][2]^2(1, 4, 2, 8, 5, 8, 2, 4, 1),$$

where $\hat{\mu}$ is the special slope function considered before. Recall that $\hat{\sigma}(\gamma) = (\epsilon, \epsilon) \cdot \gamma$ for some sufficiently small ϵ. Now we change the slope to $\tilde{\sigma} = (\epsilon, 0)$, then

$$|\operatorname{Mod}_{(\gamma,\alpha)}^{\tilde{\mu}}(A_2(K_3))| = |\mathbb{P}^2||\operatorname{Mod}_\alpha^\mu(K_3)|.$$

Application to Homological Stratification

Definition

For any representation E, the *E-homological stratification* of $\mathrm{Rep}_\alpha^\mu(Q)$ is the decomposition of $\mathrm{Rep}_\alpha^\mu(Q)$ into (finite) disjoint union of locally closed subvarieties $\mathrm{Rep}_\alpha^\mu(Q; E, h)$, where

$$\mathrm{Rep}_\alpha^\mu(Q; E, h) = \{M \in \mathrm{Rep}_\alpha^\mu(Q) \mid \hom_Q(E, M) = h\}.$$

Theorem

$|\mathrm{Rep}_\alpha^\mu(Q; E, h)|$ *can be explicitly computed from* $\mathrm{Gr}_\gamma(E)$. *When* E *is add-polynomial-count and* $\mathrm{Mod}_\alpha^\mu(Q)$ *is a geometric quotient, each homological strata on* $\mathrm{Mod}_\alpha^\mu(Q)$ *is polynomial-count.*

The proof combines our method with a wall-crossing formula of M. Reineke.

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Time for
questions and comments
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