On surface cluster algebras: Band and snake graph calculus

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Background

- Cluster algebras, introduced by **Fomin and Zelevinsky** in [FZ1] form a class of combinatorially defined commutative algebras, and the set of **generators** of a cluster algebra, **cluster variables**, is obtained by an iterative process.
- A **surface cluster algebra** $\mathcal{A}(S, M)$ is associated to a surface S with boundary that has finitely many marked points.
- Cluster variables are in bijection with certain curves [FST], called arcs.
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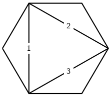

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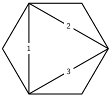

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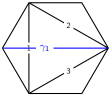

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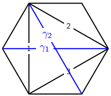

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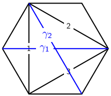

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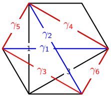

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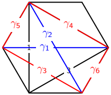

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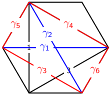

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Let $\mathcal{A}(S, M)$ cluster algebra associated to a surface (S, M).

We have the following situation:

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"How much can we recover from snake graphs themselves?"

In particular,

- When do the two arcs corresponding to two snake graphs cross?
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Abstract Snake Graphs

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A **snake graph** $\mathcal{G}$ is a connected graph in $\mathbb{R}^2$ consisting of a finite sequence of tiles $G_1, G_2, \ldots, G_d$ with $d \geq 1$, such that for each $i = 1, \ldots, d-1$

(i) G_i and G_{i+1} share exactly one edge e_i and this edge is either the north edge of G_i and the south edge of G_{i+1} or the east edge of G_i and the west edge of G_{i+1}.

(ii) G_i and G_j have no edge in common whenever $|i-j| \geq 2$.

(ii) G_i and G_j are disjoint whenever $|i-j| \geq 3$.

Example

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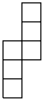

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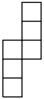

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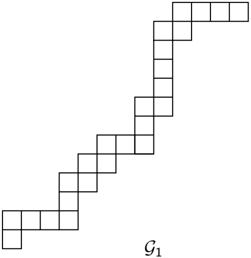

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- $\mathcal{G} = (G_1, G_2, \ldots, G_d)$
- $\mathcal{G}[i, i+t] = (G_i, G_{i+1}, \ldots, G_{i+t})$
- We denote by e_i the interior edge between the tiles G_i and G_{i+1}.

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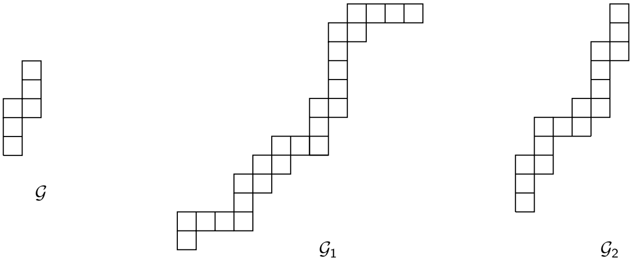

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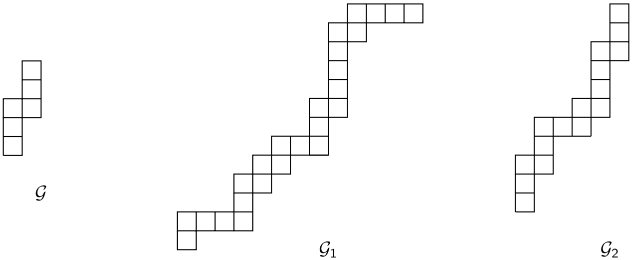

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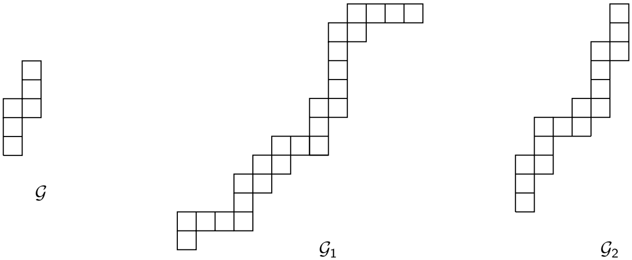

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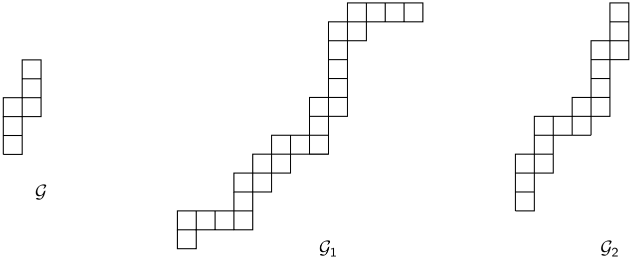

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Local Overlaps

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We say two snake graphs $\mathcal{G}_1$ and $\mathcal{G}_2$ have a **local overlap** $\mathcal{G}$ if $\mathcal{G}$ is a maximal subgraph contained in both $\mathcal{G}_1$ and $\mathcal{G}_2$.

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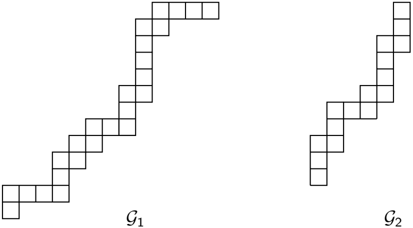

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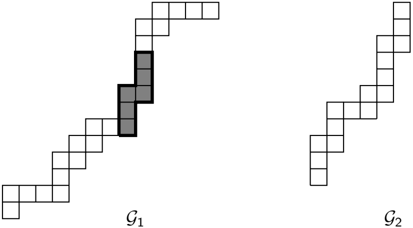

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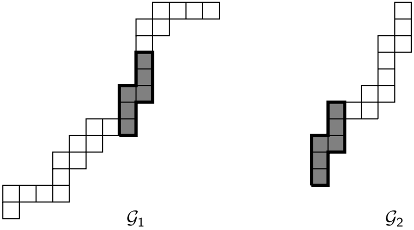

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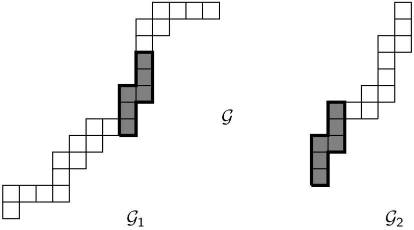

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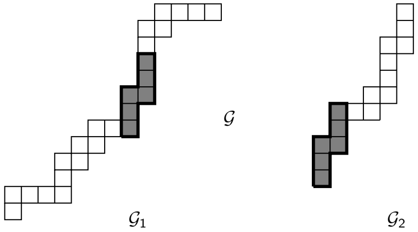

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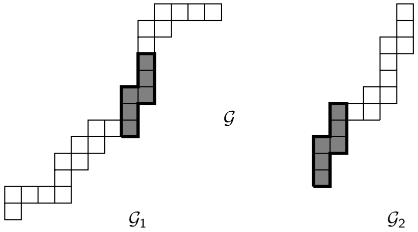

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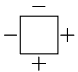

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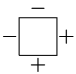

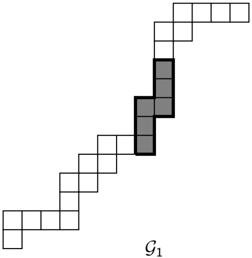

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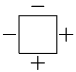

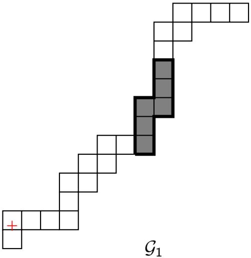

$\mathcal{G}_1$

Sign Function

Definition

A **sign function** f on a snake graph $\mathcal{G}$ is a map f from the set of edges of $\mathcal{G}$ to $\{+,-\}$ such that on every tile in $\mathcal{G}$ the north and the west edge have the same sign, the south and the east edge have the same sign and the sign on the north edge is opposite to the sign on the south edge.

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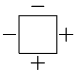

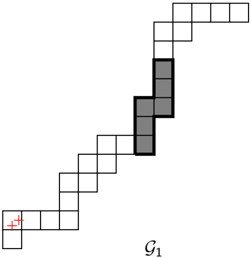

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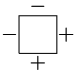

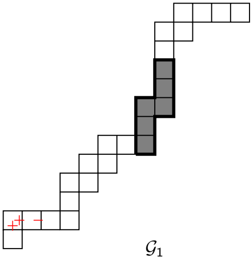

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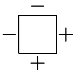

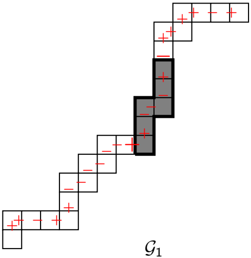

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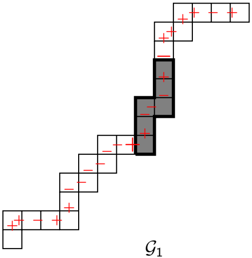

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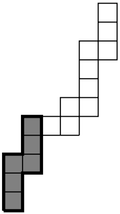

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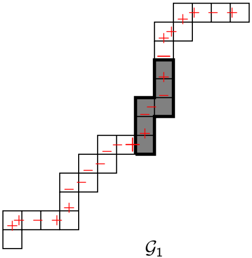

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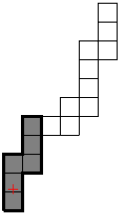

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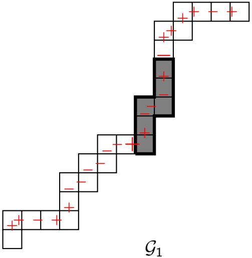

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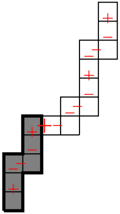

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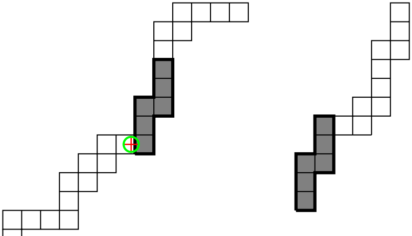

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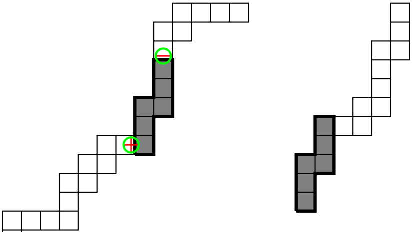

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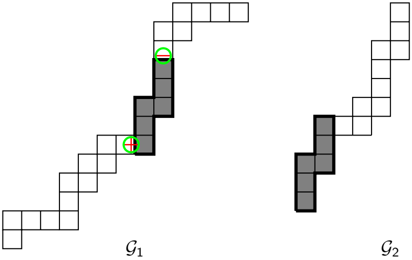

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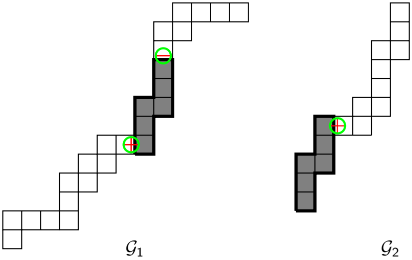
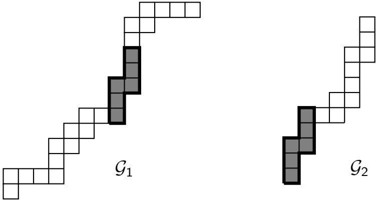


Example: Resolution $\mathrm{Res}_{\mathcal{G}}(\mathcal{G}_1, \mathcal{G}_2)$



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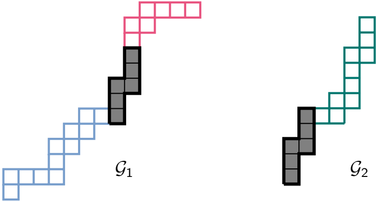

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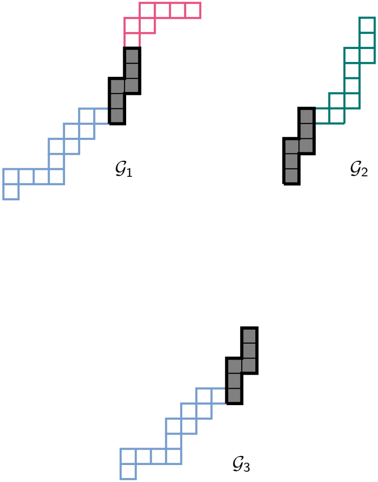

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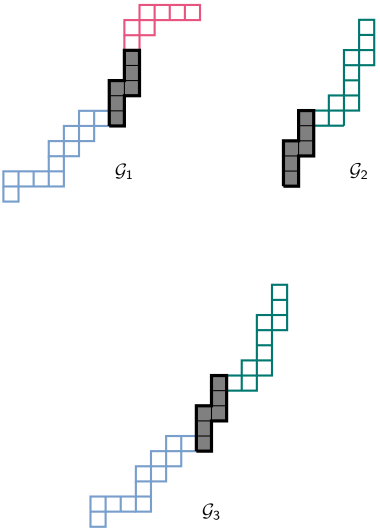

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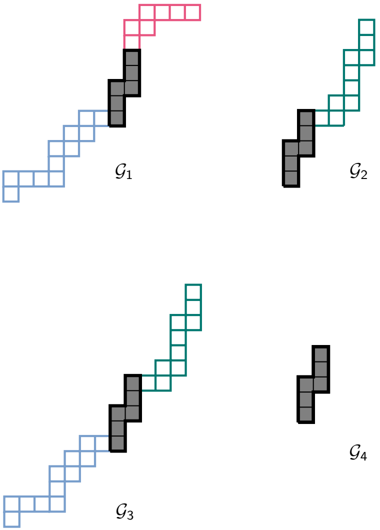

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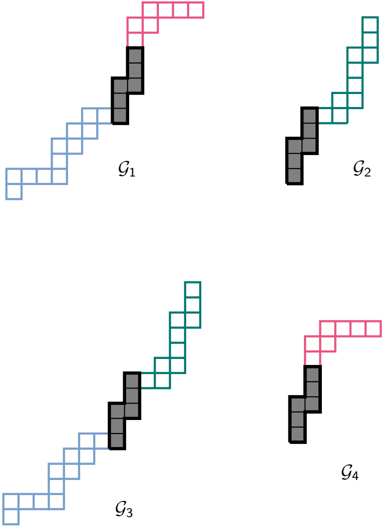

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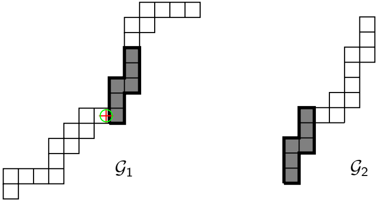

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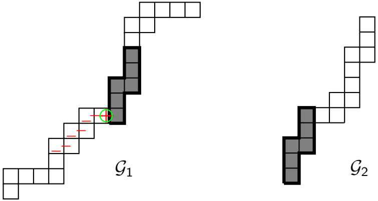

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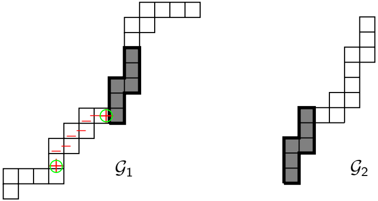

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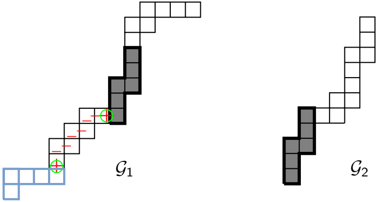
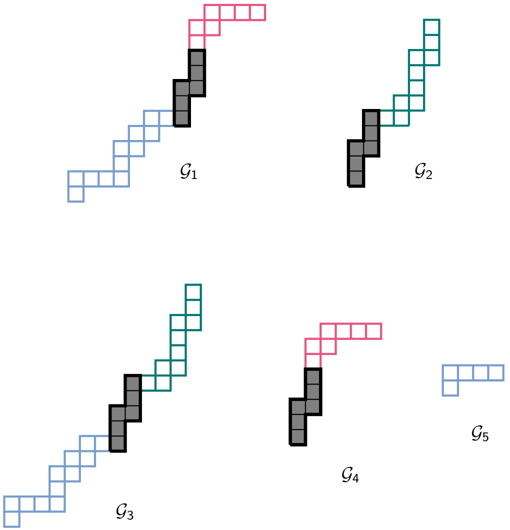


Example: Resolution (Continued)



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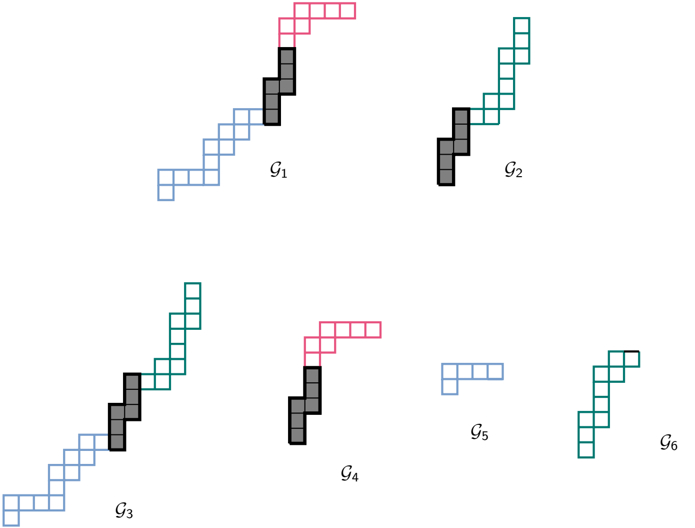

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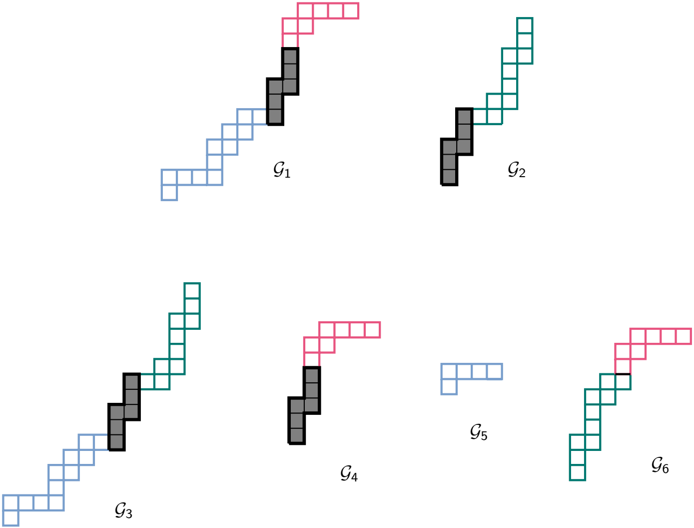

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Assumption: We will assume that $s > 1$, $t < d$, $s' = 1$ and $t' < d'$. For all other cases, see [CS].

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Band and snake graph calculus

I. Canakci, R. Schiffler

Motivation

Abstract Snake Graphs

Relation to Cluster Algebras

Band Graphs and Future Directions

Bijection of Perfect Matchings

Definition

A **perfect matching** P of a graph G is a subset of the set of edges of G such that each vertex of G is incident to exactly one edge in P.

- Let Match(G) denote the set of all perfect matchings of the graph G and Match$(\mathrm{Res}_{\mathcal{G}}(\mathcal{G}_1, \mathcal{G}_2))$ = Match$(\mathcal{G}_3 \sqcup \mathcal{G}_4) \sqcup$ Match$(\mathcal{G}_5 \sqcup \mathcal{G}_6)$

Theorem (CS)

Let $\mathcal{G}_1, \mathcal{G}_2$ be two snake graphs. Then there is a bijection

$$\mathrm{Match}\,(\mathcal{G}_1 \sqcup \mathcal{G}_2) \longrightarrow \mathrm{Match}\,(\mathrm{Res}_{\mathcal{G}}(\mathcal{G}_1, \mathcal{G}_2))$$

- Note that we construct the bijection map and its inverse map explicitly.

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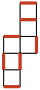

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Definition

A **perfect matching** P of a graph G is a subset of the set of edges of G such that each vertex of G is incident to exactly one edge in P.

- Let $\text{Match}(G)$ denote the set of all perfect matchings of the graph G and
$\text{Match}(\text{Res}_{\mathcal{G}}(\mathcal{G}_1, \mathcal{G}_2)) = \text{Match}(\mathcal{G}_3 \sqcup \mathcal{G}_4) \cup \text{Match}(\mathcal{G}_5 \sqcup \mathcal{G}_6)$.

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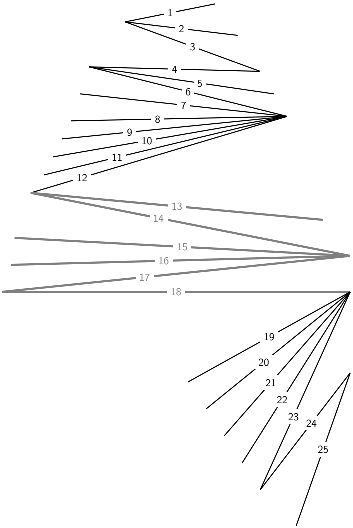

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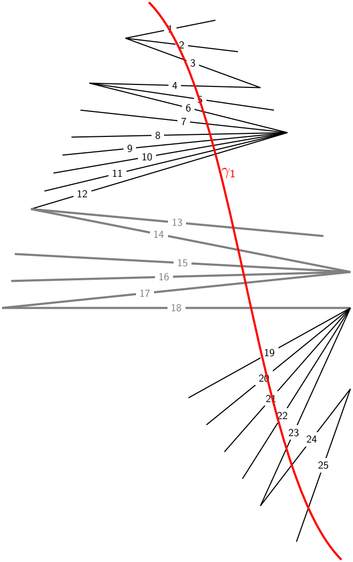

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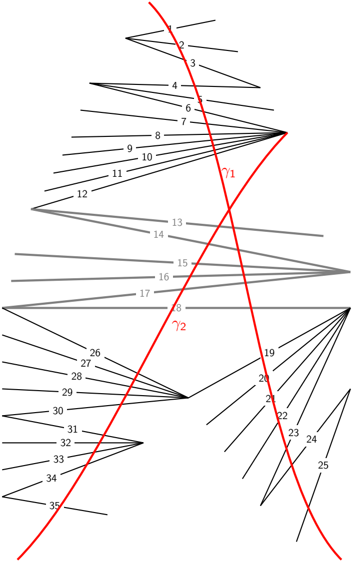

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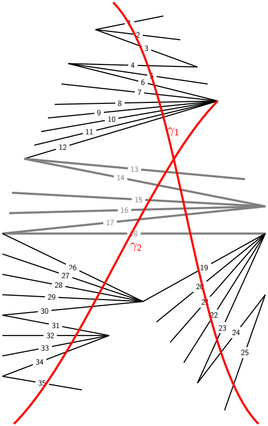

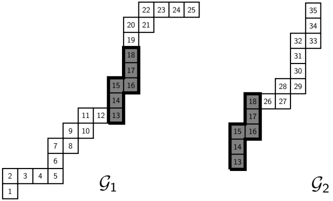

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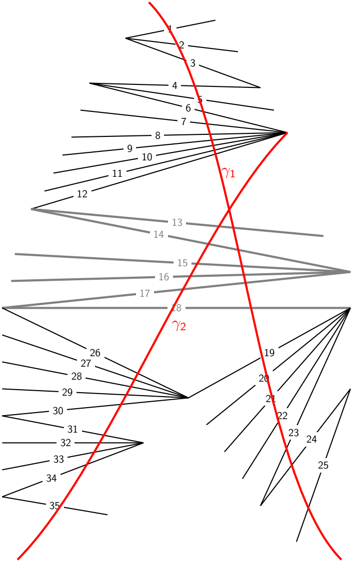

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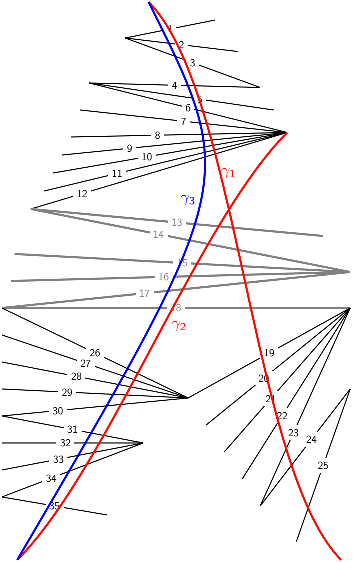

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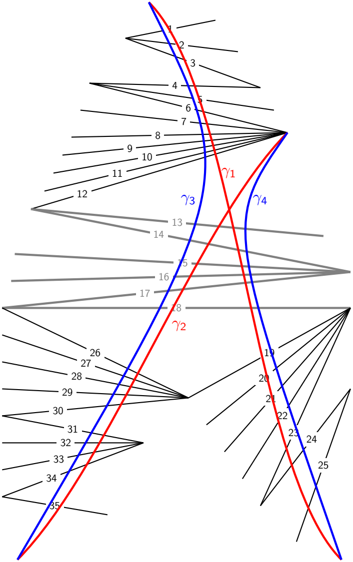

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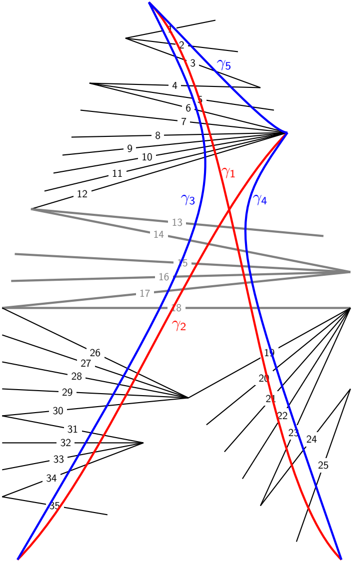

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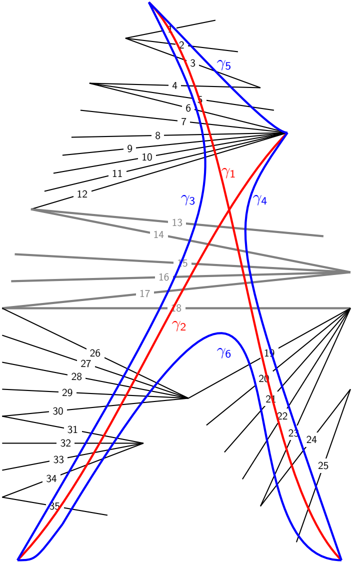

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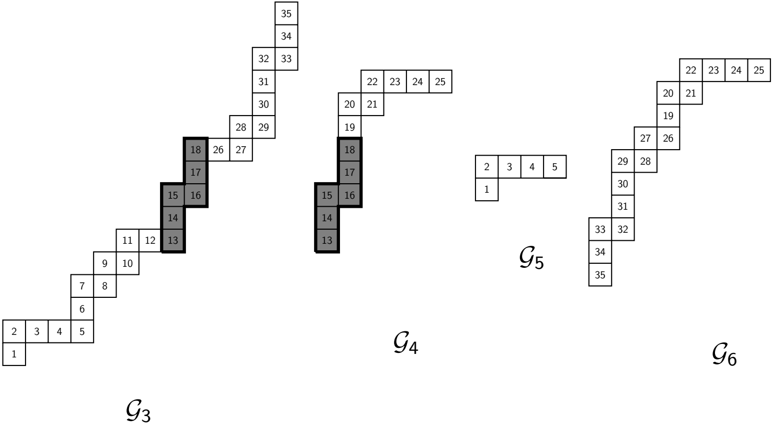

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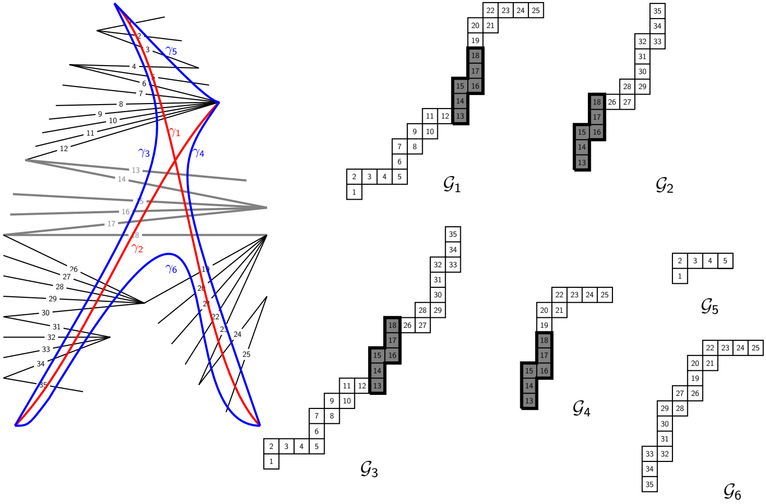

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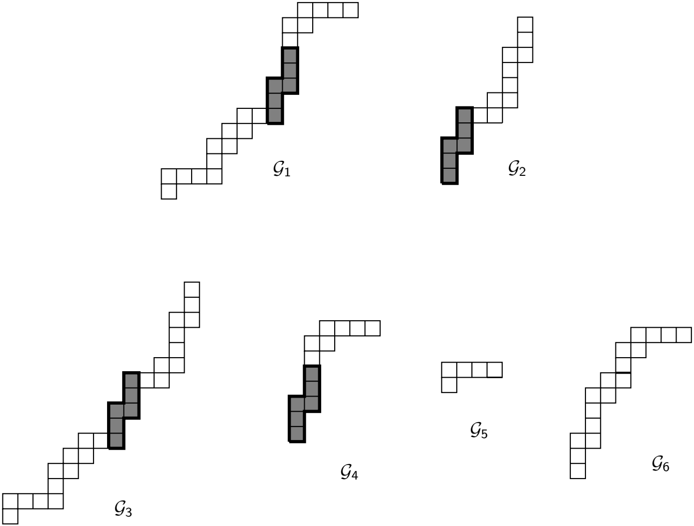

Relation to Cluster Algebras

Let γ_1 and γ_2 be two arcs and $\mathcal{G}_1$ and $\mathcal{G}_2$ their corresponding snake graphs.

Theorem (CS)

γ_1 and γ_2 **cross** *if and only if $\mathcal{G}_1$ and $\mathcal{G}_2$* **cross.**

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If γ_1 and γ_2 cross, then the snake graphs of the four arcs obtained by **smoothing the crossing** *are given by the* **resolution** $\text{Res}_{\mathcal{G}}(\mathcal{G}_1, \mathcal{G}_2)$ *of the crossing of the snake graphs $\mathcal{G}_1$ and $\mathcal{G}_2$ at the overlap $\mathcal{G}$.*

Remark

We do not assume that γ_1 and γ_2 cross only once. If the arcs cross multiple times the theorem can be used to resolve any of the crossings.

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As a corollary we obtain a new proof of the skein relations [MW].

Corollary (CS)

Let γ_1 and γ_2 be two arcs which cross and let (γ_3, γ_4) and (γ_5, γ_6) be the two pairs of arcs obtained by smoothing the crossing. Then

$$x_{\gamma_1} x_{\gamma_2} = x_{\gamma_3} x_{\gamma_4} + y(\tilde{\mathcal{G}}) x_{\gamma_5} x_{\gamma_6}$$

where $\tilde{\mathcal{G}}$ is the closure of the overlap $\mathcal{G}$.

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- Note that Musiker and Williams in [MW] use **hyperbolic geometry** to prove the skein relations.
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- Closed loops appear naturally in the process of smoothing crossings. Consider the following example.

Example

In this example we resolve two crossings of the following arcs.

Question: Is this construction straightforward?
Answer: No!

The difficulty here is to show the 'skein relations' for self-crossing arcs.

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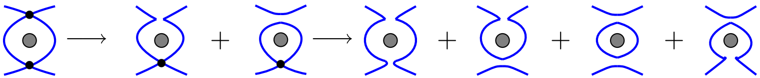

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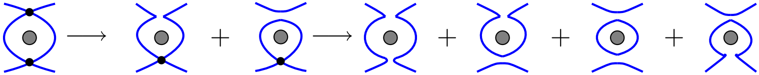

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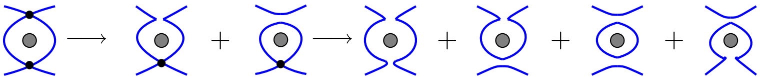

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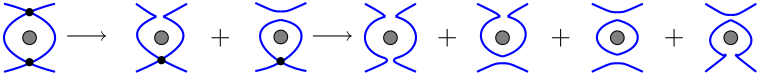

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Bibliography

- I. Canakci, R. Schiffler *Snake graph calculus and cluster algebras from surfaces*, to appear in *Journal of Algebra*, preprint available at arxiv:1209.4617v1.
- S. Fomin, M. Shapiro and D. Thurston, Cluster algebras and triangulated surfaces. Part I: Cluster complexes, *Acta Math.* **201** (2008), 83-146.
- S. Fomin and A. Zelevinsky, Cluster algebras I: Foundations, *J. Amer. Math. Soc.* **15** (2002), 497–529.
- G. Musiker, R. Schiffler and L. Williams, Positivity for cluster algebras from surfaces, *Adv. Math.* **227**, (2011), 2241–2308.
- G. Musiker, R. Schiffler and L. Williams, Bases for cluster algebras from surfaces, to appear in *Compos. Math.*
- G. Musiker and L. Williams, Matrix formulae and skein relations for cluster algebras from surfaces, preprint, `arXiv:1108.3382`.