The algebra of polynomial integro-differential operators and its group of automorphisms

V. V. Bavula

We prove that the algebra $\mathbb{I}_n := K\langle x_1, \ldots, x_n, \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}, \int_1, \ldots, \int_n\rangle$ of integro-differential operators on a polynomial algebra is a prime, central, catenary, self-dual, non-Noetherian algebra of classical Krull dimension n and of Gelfand-Kirillov dimension $2n$. Its weak dimension is n, and $n \leq \mathrm{gl.dim}(\mathbb{I}_n) \leq 2n$. All the ideals of $\mathbb{I}_n$ are found explicitly, there are only finitely many of them ($\leq 2^{2^n}$), they commute ($\mathfrak{a}\mathfrak{b} = \mathfrak{b}\mathfrak{a}$) and are idempotent ideals ($\mathfrak{a}^2 = \mathfrak{a}$). An analogue of the Hilbert's Syzygy Theorem is proved for $\mathbb{I}_n$. The group of units of the algebra $\mathbb{I}_n$ is described (it is a huge group). A canonical form is found for each integro-differential operators (by proving that the algebra $\mathbb{I}_n$ is a generalized Weyl algebra). All the mentioned results hold for the Jacobian algebra $\mathbb{A}_n$ (but $\mathrm{GK}(\mathbb{A}_n) = 3n$, note that $\mathbb{I}_n \subset \mathbb{A}_n$). It is proved that the algebras $\mathbb{I}_n$ and $\mathbb{A}_n$ are ideal equivalent.

The group G_n of automorphisms of the algebra $\mathbb{I}_n$ is found:

$$\mathrm{G}_n = S_n \ltimes \mathbb{T}^n \ltimes \mathrm{Inn}(\mathbb{I}_n) \supseteq S_n \ltimes \mathbb{T}^n \ltimes \underbrace{\mathrm{GL}_\infty(K) \ltimes \cdots \ltimes \mathrm{GL}_\infty(K)}_{2^n - 1 \text{ times}},$$

$$\mathrm{G}_1 \simeq \mathbb{T}^1 \ltimes \mathrm{GL}_\infty(K),$$

where S_n is the symmetric group, $\mathbb{T}^n$ is the n-dimensional torus, $\mathrm{Inn}(\mathbb{I}_n)$ is the group of inner automorphisms of $\mathbb{I}_n$ (which is huge). It is proved that each automorphism $\sigma \in \mathrm{G}_n$ is uniquely determined by the elements $\sigma(x_i)$'s or $\sigma(\frac{\partial}{\partial x_i})$'s or $\sigma(\int_i)$'s. The stabilizers in G_n of all the ideals of $\mathbb{I}_n$ are found, they are subgroups of *finite* index in G_n. It is shown that the group G_n has trivial centre. For each automorphism $\sigma \in \mathrm{G}_n$, an *explicit inversion formula* is given via the elements $\sigma(\frac{\partial}{\partial x_i})$ and $\sigma(\int_i)$.

Department of Pure Mathematics
University of Sheffield
Hicks Building
Sheffield S3 7RH
UK
email: v.bavula@sheffield.ac.uk