GEOMETRIC REPRESENTATION THEORY OF THE HILBERT SCHEMES
PART III
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Abstract. Realizing the fixed point basis in the equivariant cohomology of \((\mathbb{C}^2)^n\) as the Jack polynomials, we prove an equivariant version of the Lehn theorem for \(X = \mathbb{C}^2\).

1. The first Chern class of the tautological bundle

Let \(Z_n \subset X[n] \times X\) be the universal family over \(X[n]\) and \(p\) denote its projection to \(X[n]\). Then \(T_n := p_\ast \mathcal{O}(Z_n)\) is a rank \(n\) vector bundle over \(X[n]\), called the tautological bundle.\(^1\) In this section we compute the cup product operator \(c_1(T_n) \cup \bullet : H^\ast T(X[n]) \rightarrow H^\ast T(X[n])\). This operator was first studied in [L] (in the non-equivariant setting). Our exposition follows [N].

1.1. Eigenvectors of \(c_1(T_n) \cup \bullet\).

We start from a straightforward computation of \(c_1(T_n) \cup \bullet\) in the fixed point basis.

Lemma 1.1. The operator \(c_1(T_n) \cup \bullet\) is diagonalizable in the fixed point basis:

\[ c_1(T_n) \cup [\xi_\lambda] = -(n(\lambda) \epsilon_1 + n(\lambda^*) \epsilon_2)[\xi_\lambda], \]

where \(n(\lambda) := \sum (i-1) \lambda_i\).

Proof. By definition, we have \(c_1(T_n) \cup [\xi_\lambda] = c_1(T_n|_{\xi_\lambda})[\xi_\lambda]\). It remains to notice that \(c_1(T_n|_{\xi_\lambda}) = \sum_{i=1}^{(\lambda)} \sum_{j=1}^{\lambda_i} (-i-1) \epsilon_1 - (j-1) \epsilon_2 = -n(\lambda) \epsilon_1 - n(\lambda^*) \epsilon_2\).

1.2. Laplace-Beltrami operator.

Definition 1.1. The linear operator \(\Box^k_N : \Lambda^r N \rightarrow \Lambda^r N\), defined by

\[ \Box^k_N(f) = \left(\frac{k}{2} \sum_{i=1}^{N} x_i^2 \partial^2_{x_i} + \sum_{i \neq j} \frac{x_i^2}{x_i - x_j} \partial_{x_i} - r(N-1)\right), \quad f \in \Lambda^r N, \]

is called the Laplace-Beltrami operator.

Exercise 1.2. Check \(\rho_{N+1,N} \circ \Box^k_{N+1} = \Box^k_N \circ \rho_{N+1,N}\).

Hence, we can define a linear operator

\[ \Box^k : \Lambda \rightarrow \Lambda, \quad \Box^k := \lim \Box^k_N. \]

Those operators are actually diagonalizable in the basis of Jack polynomials:

Proposition 1.3. [M, Exercise VI.4.3(b)] We have: \(\Box^k(P^{(k)}_\lambda) = (n(\lambda^*)k - n(\lambda)) \cdot P^{(k)}_\lambda\).

\(^1\)The fiber of \(T_n\) at the codimension \(n\) ideal \(I \subset \mathbb{C}[x,y]\) is identified with \(\mathbb{C}[x,y]/I\). Moreover, its determinant \(\wedge^n T_n\) is actually the line bundle \(\mathcal{O}_{(\mathbb{C}^2)^n}(1)\) arising from the Proj-construction of \((\mathbb{C}^2)^n\).
1.3. Geometric interpretation of $\square^k$.

Let $\theta^T : \Lambda_\mathbb{F} \to \text{M}^T_{\text{loc}} = \oplus H^{T, BM}_i(X [n])_{\text{loc}}$ be the isomorphism from the last talk. Identifying $H^{T, BM}_i(X [n])$ with $H^{T, BM}_{i-1}(X [n])$, consider a linear operator $D : \Lambda_\mathbb{F} \to \Lambda_\mathbb{F}$ which corresponds to $c_1(T_n) \cup \bullet : H^*_{T}(X [n]) \to H^*_{T}(X [n])$ under this isomorphism.

**Theorem 1.4.** We have: $D = \epsilon_1 \cdot \square^k$.

*Proof.* According to the main result from the last time, we have:

$$\theta^T : P^{(k)}_{\Lambda} \mapsto \epsilon_1^{-|\lambda|} \epsilon_3(k)^{-1} \cdot [\xi_\lambda], \; k = -\epsilon_2/\epsilon_1.$$ 

Therefore $D$ is determined by the condition $D(P^{(k)}_{\Lambda}) = \epsilon_1(n(\lambda)^*k-n(\lambda))P^{(k)}_{\Lambda}$. Combining with Proposition 1.3, we get the result. \hfill \Box

The following is straightforward (see Appendix for the proof):

**Corollary 1.5.** Identifying $\Lambda \mathbb{C} \simeq \mathbb{C}[p_1, p_2, \ldots]$, the operator $\square^k$ is given by

$$\square^k = \frac{k}{2} \sum_{m,n>0} mn p_m + n \partial_{p_m} \partial_{p_n} + \frac{k-1}{2} \sum_{m>0} m(m-1)p_m \partial_{p_m} + \frac{1}{2} \sum_{m,n>0} (m+n)p_m p_n \partial_{p_{m+n}}.$$ 

1.4. Lehn’s formula.

In this section we reformulate Corollary 1.5 in a more standard form.

Recall that under the isomorphism $\theta^T : \Lambda_\mathbb{F} \to \text{M}^T_{\text{loc}}$, the operators $p_m$ and $-m \partial_{p_m}$ correspond to $q_{\alpha}[-m] = Z_{\alpha}[-m]$ and $q_{\alpha_1}[m] = \frac{(-1)^m}{k} Z_{\alpha_1}[m] = (-1)^{m-1} Z_{\alpha_1}[m]$, respectively. Hence, the operator $c_1(T_n) \cup \bullet$ is given by the following formula:

$$c_1(T_n) \cup \bullet = \frac{\epsilon_1 + \epsilon_2}{2} \sum_{m>0} (m-1)q_{\alpha_2}[-m]q_{\alpha_1}[m] - \sum_{m,n>0} \left( \frac{\epsilon_2}{2} q_{\alpha_2}[-m-n]q_{\alpha_1}[m]q_{\alpha_1}[n] + \frac{\epsilon_1}{2} q_{\alpha_2}[-m-n]q_{\alpha_1}[m+n] \right).$$

Let us now introduce $\delta_T : H^*_{T}(X) \to H^*_{T}(X) \otimes H^*_{T}(X)$ as the adjoint of the cup product $\cup : H^*_{T}(X) \otimes H^*_{T}(X) \to H^*_{T}(X)$ with respect to the intersection pairing. In other words, $\delta_T$ is a push-forward along the diagonal embedding $X \to X \times X$. This is a $H^*_{T}(pt)$-linear map with $\delta_T(1) = 1 \otimes [X] = \epsilon_1 \epsilon_2 \cdot 1 \otimes 1$. Iterating $\delta_T$, we get $\delta_T^2(1) = (\epsilon_1 \epsilon_2)^2 \cdot 1 \otimes \cdots \otimes 1$.

For $\alpha \in H^*_{T}(X)$ with $\delta_T(\alpha) = \sum c_i^1 \otimes c_i^2$, we set:

$$(q_m q_n)(\alpha) := \sum q_{\alpha_1}^{-}[m]q_{\alpha_2}^{[n]}.$$ 

Using this notation together with $K_X = -\epsilon_1 - \epsilon_2$ (Ker2 is generated by $dx \wedge dy$), we get:

**Theorem 1.6.** [L] We have

$$c_1(T_n) \cup \bullet = -\frac{1}{6} \sum_{m_1 + m_2 + m_3 = 0} : q_{m_1} q_{m_2} q_{m_3} : (1) - \frac{1}{4} \sum_{m} (|m| - 1) : q_m q_m : (K_X),$$

where $:$ denotes the normal ordering.

This beautiful result was first proved by Lehn ([L]) in the non-equivariant setting for any $X$. The key observation of [L] was a geometric action of Vir on $M$ discussed in the next section.
1.5. Virasoro action on $M$.

Let us first introduce another important Lie algebra:

**Definition 1.2.** The complex Lie algebra $\text{Vir}$ with a basis $\{L_n, n \in \mathbb{Z}, c\}$ and a Lie bracket

$$[L_n, L_m] = (n - m)L_{n+m} + \frac{c}{12}(n^3 - n)\delta_{n+m}^0, \quad [c, L_n] = 0, \quad n, m \in \mathbb{Z},$$

is called the Virasoro algebra. Its representation $V$ is of central charge $c_0 \in \mathbb{C}$ if $c_V = c_0 \cdot \text{Id}_V$.

Define operators $\mathcal{L}_n : H^*(X) \to \text{End}(M)$ by $\mathcal{L}_n(\alpha) := \frac{1}{2} \sum_{t \in \mathbb{Z}} : q_t q_{n-t} : (\alpha)$. According to [L, Theorem 3.3], those operators satisfy the following commutator relation:

$$[\mathcal{L}_n(\alpha), \mathcal{L}_m(\beta)] = (n - m)\mathcal{L}_{n+m}(\alpha \cup \beta) - \frac{n^3 - n}{12} \delta_{n+m}^0 \cdot (\mathcal{L}_2(\chi), \chi) \cdot \text{Id}_M.$$

**Corollary 1.7.** The operators $\{\mathcal{L}_n(1)\}$ define an action of the Virasoro algebra $Vir$ on $M$ of central charge $-c(X)$ ($c(X)$ is the Euler number of $X$).

**Remark 1.1.** This result can be considered as a slight update of the classical Vir-action on the Fock space over the Heisenberg algebra $\hat{\mathcal{H}}$ (see [KR, Proposition 2.3]).

In [L], Theorem 1.6 is derived from the following commutator formula:

$$[c_1(\mathcal{F}_n) \cup \bullet, q_{\alpha}[n]] = n \cdot \mathcal{L}_n(\alpha) + \frac{n(n-1)}{2} q_{KX \cup \alpha}[n].$$

We refer the reader to [L] for more details on this elegant result.

**Appendix A. Proof of Corollary 1.5**

In this section we prove Corollary 1.5. Let us first introduce another important Lie algebra:

$$\Box^k = \frac{k}{2} \sum_{m,n>0} mmp_{m+n}p_m p_n + \frac{k-1}{2} \sum_{m>0} m(m-1)p_m^2 + \frac{1}{2} \sum_{m,n>0} (m+n)p_m p_n p_{m+n}.$$
References


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