INVARIANT THEORY, HW5

Problem 1, 5pts. As in HW3, consider a faithful action of a torus T on a vector space V. Recall that V has an eigen-basis $v_1, \ldots, v_n$, let $\chi_1, \ldots, \chi_n$ be the corresponding eigen-characters of T. Let $\tilde{T} \subset \mathrm{GL}(V)$ be the maximal torus of all operators diagonal in the basis $v_1, \ldots, v_n$. So $T \subset \tilde{T}$. Pick a character θ of T.

a, 2pts) Show that $V^{\theta-ss} \subset V$ is $\tilde{T}$-stable, $\tilde{T}/T$ acts on $V//^{\theta}T$ in such a way that $\pi^{\theta} : V^{\theta-ss} \to V//^{\theta}T$ is $\tilde{T}$-equivariant.

b, 3pts) Show that the fixed points of $\tilde{T}/T$ on $V//^{\theta}T$ are in bijection with the subsets $I \subset \{1, \ldots, n\}$ satisfying the following two conditions

- $\chi_i, i \in I$, are linearly independent,
- and there are rational numbers $n_i, i \in I$, such that $\theta = \sum_i n_i \chi_i$ and $n_i < 0$ for all $i \in I$.

Hint: b) These stable points correspond to closed T-orbits in $V^{\theta-ss}$ that are also $\tilde{T}$-orbits.

Problem 2, 5pts. Let G be a connected factorial reductive algebraic group (i.e., $\mathbb{C}[G]$ is a UFD), let H be an algebraic subgroup of G. Note that we have the restriction map $\rho : \mathfrak{X}(G) \to \mathfrak{X}(H)$ between the character groups. Prove that $\mathrm{Pic}(G/H) \cong \operatorname{coker} \rho$.

Hint: Observe that $\mathfrak{X}(H)$ is identified $\mathrm{Pic}^G(G/H)$, the Pickard group of G-equivariant line bundles on G/H. You can also use the fact mentioned in Lecture 18 that every G-equivariant structure on the structure sheaf of a normal variety is given by a character of G.

Problem 3, 5pts. Let X be an affine algebraic variety equipped with an action of a reductive algebraic group G. Show that there are finitely many reductive subgroups $H_1, \ldots, H_k \subset G$ such that every closed G-orbit in X is G-equivariantly isomorphic to one of G/H_i.

Hint: Reduce to the case of a vector space. Choose a point x with a closed orbit and let S be an etale slice through x and $H := G_x$. Then the stabilizers in the image of $G \times^H S$ in X are conjugate to the stabilizers for the action of H on S.