INVARIANT THEORY, HINTS FOR HW3

Problem 1, 7pts. This problem describes the algebra of invariants $\mathbb{C}[V]^T$ under a faithful action of a torus T on a vector space V. Recall that V has an eigen-basis $v_1, \ldots, v_n$, let $\chi_1, \ldots, \chi_n$ be the corresponding eigen-characters of T. Let $\tilde{T} \subset \mathrm{GL}(V)$ be the maximal torus of all operators diagonal in the basis $v_1, \ldots, v_n$. So $T \subset \tilde{T}$. Note that $\tilde{T}/T$ is a torus naturally acting on $\mathbb{C}[V]^T$ and the character lattice $\mathfrak{X}(\tilde{T}/T)$ naturally embeds into $\mathfrak{X}(\tilde{T}) = \mathbb{Z}^n$. Let $\mathcal{M}$ denote the submonoid $\{(m_1, \ldots, m_n) \in \mathbb{Z}^n_{\geqslant 0} | \sum_{i=1}^n m_i \chi_i = 0\} \subset \mathbb{Z}^n_{\geqslant 0}$.

a, 3pts) Show that each eigenspace of $\tilde{T}/T$ in $\mathbb{C}[V]^T$ has dimension 0 or 1 and the dimension is equal to 1 iff the corresponding eigencharacter lies in $\mathcal{M}$. For $\psi \in \mathcal{M}$ let f_ψ denote an eigen-vector for ψ in $\mathbb{C}[V]^T$.

b, 2pts) Show that elements $\psi_1, \ldots, \psi_k$ generate the monoid $\mathcal{M}$ iff the polynomials $f_{\psi_1}, \ldots, f_{\psi_k}$ generate the algebra $\mathbb{C}[V]^T$. Deduce that $\mathcal{M}$ is a finitely generated monoid.

c, 2pts) Show that the relations of the form $f_{\psi_1}^{s_1} \ldots f_{\psi_k}^{s_k} - f_{\psi_1}^{r_1} \ldots f_{\psi_k}^{r_k} = 0$ with $s_i, r_i \geqslant 0$ and $\sum_{i=1}^k (s_i - r_i)\psi_i = 0$ generate the ideal of relations between the generators $f_{\psi_1}, \ldots, f_{\psi_r}$.[1]

Hints:

a) We know a spanning set for $\mathbb{C}[V]^T$.

b) Use that $\mathbb{C}[V]^T$ *has no zero divisors.*

c) Consider the algebra $\mathbb{C}[x_1, \ldots, x_r]$ *with an epimorphism* $\mathbb{C}[x_1, \ldots, x_r] \twoheadrightarrow \mathbb{C}[V]^T$. *Equip* $\mathbb{C}[x_1, \ldots, x_r]$ *with a suitable* $\tilde{T}/T$*-action and consider a suitable ideal. What are the dimensions of* $\tilde{T}/T$*-eigen-spaces in the quotient?*

Problem 2, 7pts. This problem studies the subvarieties of nilpotent elements (=elements whose orbit closures contain 0) for natural actions of classical groups. Use the Hilbert-Mumford theorem[2] to prove the following claims.

a, 2pts) Let $G = \mathrm{GL}(U)$ act on $V := U^{\oplus k} \oplus U^{*\oplus \ell}$ in a natural way. Show that $(u_1, \ldots, u_k, u^1, \ldots, u^l)$ is nilpotent iff $\langle u_i, u^j \rangle = 0$ for all i, j.

[1] Of course, there is a more economical set of relations.

[2] Solutions based on computing the algebras of invariants do not count

b) 2pts) Let $G = \mathrm{SL}(U)$ acts on $U^{\oplus k}$. Then $(u_1, \dots, u_k)$ is nilpotent iff $u_1, \dots, u_k$ do not span U.

c, 3pts) Let U be a finite dimensional space equipped with an orthogonal form $(\cdot, \cdot)$. Consider the natural action of the orthogonal group $G = \mathrm{O}(U)$ on $V = U^{\oplus k}$. Then $(u_1, \dots, u_k)$ is nilpotent iff $(u_i, u_j) = 0$ for all i, j. State and prove the corresponding claim for the symplectic group.

Hints: a and b) Deal with diagonal one-parameter subgroups first.

c) Note that the sum of the eigen-spaces for $\nu : \mathbb{C}^\times \to G$ corresponding to the positive powers of t is isotropic. Also all isotropic subspaces are conjugate to a subspace in a fixed maximal isotropic subspace.