HINTS FOR HOMEWORK 2

Problem 1 (5pts). Let G_1, G_2 be algebraic groups such that $G_1 = \mathbb{G}_a^n$ and the connected component G_2° is a torus. Show that there are no nontrivial algebraic group homomorphisms between G_1, G_2 in either direction.

Hint: Show that there are no algebraic variety morphisms from G_1 to G_2. To show that there are no algebraic group homomorphisms from G_2 to G_1 look at finite order elements.

Problem 2 (4pts). Let H be a connected algebraic group and Z be a finite normal subgroup of H. Show that Z is in the center of H.

Hint: Look at the conjugation action of H on Z.

Problem 3 (4pts). Let G be a semisimple algebraic group, $\mathfrak{g}$ its Lie algebra and $x \in \mathfrak{g}$. Assume that the centralizer $Z_G(x)$ is reductive. Show that x is semisimple. You are allowed to use facts proved in Lecture 9.

Hint: Reduce to the case when x is nilpotent. Then you could get an inspiration from the proof of Kostant's theorem in Lecture 9 (that for two $\mathfrak{sl}_2$-triples $(e,h,f),(e,h',f')$ there is $g \in Z_G(e)$ mapping h to h'). Or you could use the fact that the centralizer of a reductive subgroup in a reductive group is reductive.

Extra-credit problem. This problem explains the classification of nilpotent orbits in the classical Lie algebras $\mathfrak{so}_n, \mathfrak{sp}_n$ (in the latter case n is even, of course).

a) Show that the nilpotent O_n-orbits in $\mathfrak{so}_n$ and the nilpotent Sp_n-orbits in $\mathfrak{sp}_n$ are uniquely recovered form their Jordan types (a partition of n).

b) The partitions appearing for $\mathfrak{so}_n$ (resp., $\mathfrak{sp}_n$) are precisely those where the multiplicity of every even (resp., odd) part is even.

c) Show that a nilpotent O_n-orbit splits into two SO_n-orbits if and only if the parts of the corresponding partition are all even.