

HIGHEST WEIGHT CATEGORIES

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Let \mathbb{F} be a field, Λ be a finite poset and \mathcal{C} be an \mathbb{F} -linear artinian category with simple objects parameterized by Λ . By a highest weight category structure on \mathcal{C} with respect to Λ we mean a collection of *standard* objects $\Delta(\lambda)$ in \mathcal{C} , one for each $\lambda \in \Lambda$ such that

- $\text{Hom}(\Delta(\lambda), \Delta(\mu)) \neq 0 \Rightarrow \lambda \leq \mu$.
- $\text{End}(\Delta(\lambda)) = \mathbb{F}$ for any λ .
- \mathcal{C} has enough projectives. The indecomposable projectives are parameterized by Λ , $\lambda \mapsto P(\lambda)$, where $P(\lambda)$ is the projective cover of $L(\lambda)$. The object $P(\lambda)$ surjects onto $\Delta(\lambda)$ and the kernel admits a filtration whose quotients are of the form $\Delta(\mu)$ with $\mu > \lambda$.

We write \mathcal{C}^Δ for the full subcategory of all standardly filtered objects in \mathcal{C} , i.e., all objects that admit a filtration with standard quotients.

0.1. Simple constituents of standards. Let $L(\lambda)$ denote the simple corresponding to $\lambda \in \Lambda$. Show that if $L(\lambda)$ occurs in $\Delta(\mu)$, then $\lambda \leq \mu$. Moreover, show that the multiplicity of $L(\lambda)$ in $\Delta(\lambda)$ is 1, and $\Delta(\lambda) \twoheadrightarrow L(\lambda)$.

0.2. Ext's between standards. Show that if $\text{Ext}^i(\Delta(\lambda), \Delta(\mu)) \neq 0$ for some $i > 0$, then $\lambda < \mu$.

0.3. Subcategories. Let $\Lambda' \subset \Lambda$ be an ideal in the sense that if $\lambda \in \Lambda'$ and $\mu \leq \lambda$, then $\mu \in \Lambda'$. Consider the Serre subcategory $\mathcal{C}' \subset \mathcal{C}$ spanned by the simples $L(\lambda)$ with $\lambda \in \Lambda'$.

- (1) Show that \mathcal{C}' is a highest weight category with standard objects $\Delta(\lambda)$, $\lambda \in \Lambda'$.
- (2) Show that the left adjoint functor $\iota^!$ to the embedding $\mathcal{C}' \hookrightarrow \mathcal{C}$ is exact on \mathcal{C}^Δ .
- (3) Deduce that there is a projective resolution P^\bullet in \mathcal{C} of $M \in \mathcal{C}'^\Delta$ such that $\iota^!(P^\bullet)$ is a projective resolution of M in \mathcal{C}' (in fact, this is true for any projective resolution).

0.4. Quotients. Now let $\mathcal{C}'' := \mathcal{C}/\mathcal{C}'$ and π be the quotient functor $\mathcal{C} \rightarrow \mathcal{C}''$. Let $\pi^!$ denote its left adjoint functor.

- (1) Show that \mathcal{C}'' is a highest weight category with standard objects $\pi(\Delta(\lambda))$.
- (2) Moreover, show that the natural morphism $\pi^!(\pi(M)) \rightarrow M$ is an isomorphism provided M admits a filtration whose quotients are $\Delta(\mu)$ with $\mu \notin \Lambda'$.
- (3) Deduce that $\pi^!$ gives rise to an equivalence between \mathcal{C}''^Δ and a full subcategory in \mathcal{C}^Δ consisting of all objects whose filtration quotients are $\Delta(\mu)$ with $\mu \notin \Lambda'$.

0.5. Characterization of projectives. Show that a standardly filtered object $P \in \mathcal{C}$ is projective if and only if $\text{Ext}^1(P, \Delta(\lambda)) = 0$ for all $\lambda \in \Lambda$.

0.6. Axiomatic characterization of standards. Show that the objects $\Delta(\lambda)$ are uniquely recovered from the poset structure on Λ as follows. Pick $\lambda \in \Lambda$. Let $\Lambda_{\leq \lambda} = \{\mu \in \Lambda \mid \mu \leq \lambda\}$. Consider the Serre subcategory $\mathcal{C}_{\leq \lambda}$ spanned by the simples $L(\mu)$ with $\mu \in \Lambda_{\leq \lambda}$. Then $\Delta(\lambda)$ is the projective cover of $L(\lambda)$ in $\mathcal{C}_{\leq \lambda}$.

0.7. **Costandard objects.** Let $\nabla(\lambda)$ stand for the injective hull of $L(\lambda)$ in $\mathcal{C}_{\leq \lambda}$. This is a so called *costandard object*. Show that $\dim \operatorname{Hom}(\Delta(\lambda), \nabla(\mu)) = \delta_{\lambda, \mu}$, while $\operatorname{Ext}^i(\Delta(\lambda), \nabla(\mu)) = 0$ for $i > 0$.

0.8. **Highest weight structure on \mathcal{C}^{opp} .** Prove that the collection of costandard objects $\nabla(\lambda)$ makes the opposite category \mathcal{C}^{opp} into a highest weight category with respect to the poset Λ .

0.9. **Characterization of (co)standardly filtered objects.** Prove the object $M \in \mathcal{C}$ is Δ -filtered if and only if $\operatorname{Ext}^i(M, \nabla(\lambda)) = 0$ for all $\lambda \in \Lambda, i > 0$, if and only if $\operatorname{Ext}^1(M, \nabla(\lambda)) = 0$ for all $\lambda \in \Lambda$. Deduce the kernel of an epimorphism of standardly filtered objects is standardly filtered. State and prove the dual statements.

0.10. **BGG reciprocity.** Show that the multiplicity of $\Delta(\lambda)$ in $P(\mu)$ coincides with the multiplicity of $L(\mu)$ in $\nabla(\lambda)$.

0.11. **Tilting objects.** An object in \mathcal{C} is called *tilting* if it is both standardly and costandardly filtered.

- (1) For $\lambda \in \Lambda$ consider an object $T(\lambda)$ constructed as follows. Order linearly elements of $\{\mu \in \Lambda \mid \mu \leq \lambda\}$ refining the original poset structure, say $\lambda = \lambda_1 > \lambda_2 > \dots > \lambda_k$. Construct the object $T^i(\lambda), i = 1, \dots, k$ inductively as follows. Set $T^1(\lambda) = \Delta(\lambda)$. Further, if $T^{i-1}(\lambda)$ is already defined let $T^i(\lambda)$ be the extension of $\operatorname{Ext}^1(\Delta(\lambda_i), T^{i-1}(\lambda)) \otimes \Delta(\lambda_i)$ by $T^{i-1}(\lambda)$ corresponding to the unit endomorphism of $\operatorname{Ext}^1(\Delta(\lambda_i), T^{i-1}(\lambda))$. Show that $T(\lambda) := T^k(\lambda)$ is an indecomposable tilting.
- (2) Prove that any other tilting in \mathcal{C} is isomorphic to the direct sum of the objects $T(\lambda)$.

0.12. **Ringel duality.** Set $T := \bigoplus_{\lambda \in \Lambda} T(\lambda)$. Let \mathcal{C}^\vee be the category of finitely generated $\operatorname{End}(T)$ -modules. Show that this category is highest weight with respect to the opposite poset Λ^{opp} with standard objects $\operatorname{Hom}(\Delta(\lambda), T)$. Show that $(\mathcal{C}^\vee)^\Delta \cong (\mathcal{C}^\Delta)^{opp}$ and that, under this identification, the projective objects in \mathcal{C} correspond to tilting objects in \mathcal{C}^\vee , while tilting objects in \mathcal{C} correspond to projective objects in \mathcal{C}^\vee . Finally, identify $(\mathcal{C}^\vee)^\vee$ with \mathcal{C}^{opp} .